The fractal geometry of Brownian motion

by

Paul Potgieter

THE FRACTAL GEOMETRY OF BROWNIAN MOTION

by

PAUL POTGIETER

submitted in accordance with the requirements for the degree of

DOCTOR OF PHILOSOPHY

in the subject

MATHEMATICS

at the

UNIVERSITY OF SOUTH AFRICA

PROMOTER: PROF W L FOUCHÉ

30 NOVEMBER 2004

The fractal geometry of Brownian motion

by P Potgieter

Degree: Doctor of Philosophy

Subject: Mathematics

Promoter: Prof W L Fouché

Summary

After an introduction to Brownian motion, Hausdorff dimension, nonstandard analysis and Loeb measure theory, we explore the notion of a nonstandard formulation of Hausdorff dimension. By considering an adapted form of the counting measure formulation of Lebesgue measure, we find that Hausdorff dimension can be computed through a counting argument rather than the traditional way. This formulation is then applied to obtain simple proofs of some of the dimensional properties of Brownian motion, such as the doubling of the dimension of a set of dimension smaller than 1/2 under Brownian motion, by utilising Anderson's formulation of Brownian motion as a hyperfinite random walk. We also use the technique to refine a theorem of Orey and Taylor's on the Hausdorff dimension of the rapid points of Brownian motion. The result is somewhat stronger than the original. Lastly, we give a corrected proof of Kaufman's result that the rapid points of Brownian motion have similar Hausdorff and Fourier dimensions, implying that they constitute a Salem set.

Contents

Preface

This thesis grew from a study of Brownian motion and its relation to descriptive complexity, initially undertaken by my supervisor, Willem Fouché [11]. I came into contact with the fractal geometric aspects of Brownian motion through a lecture series given by Prof. Fouché. Also inspired by the book of Kahane [21] and the classic paper of Orey and Taylor [31], I undertook a study of the subject, specifically regarding the rapid points of Brownian motion, the points of exceptional growth on a Brownian path. After familiarising myself with the basics of nonstandard analysis and Loeb measure theory (at the suggestion of my supervisor) and unaware of any work relating Hausdorff dimension and nonstandard analysis, I found a very simple and intuitive method of approaching certain dimensional problems through hyperfinite counting arguments. By using Anderson's construction of Brownian motion on a hyperfinite time line [7], I found that these methods were suitable for providing a clear explanation of the dimensional behaviour of certain sets under Brownian motion.

The rapid points of Brownian motion turn out to have other interesting and more subtle properties, specifically regarding Fourier dimension. This considers the asymptotic behaviour of Fourier transforms of measures on exceptional sets. These properties have been intensively investigated by Kahane [21] and Kaufman [22], but the subject is by no means exhausted. In fact, many fundamental questions remain unanswered. For instance, the Fourier dimension of the zero set of Brownian motion is unknown, even though its Hausdorff dimension was calculated almost 50 years ago. It is even unknown whether it is a set of multiplicity.

This thesis starts with a basic introduction to Brownian motion and Hausdorff dimension, with a more thorough historical sketch being provided of the latter. Although the history of Brownian motion makes fascinating reading in itself, I felt that focusing on Hausdorff dimension provides a good segue into a discussion of nonstandard analysis and Loeb measure theory, which comprises the second chapter. This part, especially, is heavily indebted to Cutland's very clear survey of the subject [7].

The third chapter deals with Hausdorff dimension in a nonstandard context, initially inspired by the construction of Lebesgue measure as a counting measure utilising Loeb measure theory. I have only recently become aware of the treatment of the subject by Wattenberg [36], though our approaches are somewhat different. It is to be expected that the results in this chapter could also be expanded to Hausdorff measure with respect to arbitrary functions, but this was not necessary in the context of this thesis. The chapter also discusses capacity and Hausdorff dimension and provides a nonstandard version of Frostman's lemma. This provides an initial glimpse of Chapter 6, since it is the first encounter in this thesis of the behaviour of Fourier transforms of measures on exceptional sets.

The fourth chapter applies the previous three to the study of the fractal geometry of Brownian motion. Some of the well-known fractal properties of Brownian motion are mentioned and some are proved using nonstandard notions. Although none of the results in this section are new, to my knowledge this is the first time they have been proved using nonstandard methods. In light of Anderson's construction of Brownian motion, the results attain a certain intuitive clarity.

The paper of Orey and Taylor [31] was the major inspiration for the fifth chapter. Although their result is a consequence of the results in the chapter, I provide a more constructive proof, dealing with certain properties of covering the set of rapid points of Brownian motion with dyadic intervals. (This part is also indebted to Kaufman's treatment of Fourier dimension, to be discussed in Chapter 6.) The nature of the proof lends it applicability to the study of rapid points of complex oscillations (for some initial results in this regard, see [11]). This theme will be pursued in an upcoming paper with Fouché.

The difficult and subtle study of the Fourier dimension of the rapid points is the subject of Chapter 6. It is shown that the Hausdorff and Fourier dimensions of the set of rapid points of Brownian motion are equal, implying that they form a so-called Salem set. Raphael Salem first constructed a random set with this property [34]. Kaufman later constructed a deterministic Salem set, a construction which was clarified in 1996 by Bluhm [5]. This chapter is essentially a reworking of Kaufman's original [22] proof that the rapid points of Brownian motion form a Salem set. I felt that not all Kaufman's conclusions are entirely supported in his original paper, although the fundamental ideas are sound. Here I endeavoured to provide his argument with the necessary mathematical rigour and to clarify the underlying intuition.

This thesis is merely an introduction to a fascinating and important mathematical investigation. The idea of a Salem set is not yet fully understood, but hopefully the study of Brownian motions and their constructive counterparts, complex oscillations, will yield greater understanding.

1 Introduction to Brownian motion and Hausdorff dimension

1.1 Properties of Brownian motion

Brownian motion is a phenomenon with an essentially simple definition, but which possesses varied and surprising properties. It is well known that the name derives from the biologist Robert Brown, who observed the motion of a grain of pollen suspended in a drop of water in 1828. Although this phenomenon had been observed well before 1800, Brown's contribution was his conclusion that the process was not biological, but physical in nature. Although Brown is not otherwise famous in physics or mathematics, he made many contributions to biology, and his biographical entry in the *Encyclopaedia Britannicas* of 1878 did not even mention Brownian motion. Further illustrious names associated with the development of Brownian motion are Bachelier, Perrin and Einstein. The true theory of mathematical (as opposed to physical) Brownian motion began with Wiener, who in 1923 defined Brownian motion in the space of continuous functions [37].

In this thesis we will be interested in those properties of the sample paths (trajectories) that can be described in terms of Hausdorff and Fourier dimensions and specifically sets where those are the same, or the so-called Salem sets. We start with an introduction to Brownian motion, not a comprehensive one by any means, but one which highlights the properties we will use for our investigation.

Definition 1.1. *Given a probability space* $(\Omega, \mathcal{B}, \mathbf{P})$*, a* Brownian motion *is a stochastic process* X *from* $\Omega \times [0,1]$ *to* $\mathbb{R}$ *satisfying the following properties:*

1. *Each path* $X(\omega, \cdot) : [0,1] \to \mathbb{R}$ *is almost surely continuous*
2. $X(\omega, 0) = 0$ *almost surely*
3. *For* $0 \le t_1 < t_2 \cdots < t_n \le 1$*, the random variables* $X(\omega, t_1), X(\omega, t_2) - X(\omega, t_1), \ldots, X(\omega, t_n) - X(\omega, t_{n-1})$ *are independent and normally distributed with mean* 0 *and variance* $t_1, t_2 - t_1, \ldots, t_n - t_{n-1}$.

This means that for $0 \le t_1 < t_2 < \cdots < t_n \le 1$ and if the sets $A_1, A_2, \ldots, A_n$ are Borel subsets of the reals, the probability of the event

$$\{\omega \in \Omega : (X(t_1), \ldots, X(t_n)) \in A_1 \times \cdots \times A_n\} \tag{1.1}$$

is given by

$$\int_{A_1} \cdots \int_{A_n} \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \exp\left[\frac{-(y_j - y_{j-1})^2}{2(t_j - t_{j-1})}\right] dy_n \ldots dy_1, \tag{1.2}$$

where we have set $t_0 = y_0 = 0$.

Note that we will from now on denote a Brownian path $X(\omega, t)$ by $X(t)$.

To explore this definition in a familiar context we turn to $C[0,1]$, the set of real-valued continuous functions on the unit interval. We denote by Σ the Borel σ-algebra of subsets of $C[0,1]$, where $C[0,1]$ has the uniform norm topology. We will now show how to construct Brownian motion on this space in a way which captures the notion that Brownian motion may sometimes be viewed as a limit of random walks. A similar construction will be used in Chapter 3 to find a hyperfinite version of Brownian motion.

We first state the Central Limit Theorem [4], since we will later present a theorem of Donsker, which is an extension of it.

Theorem 1.1. *Let $\{X_j : j \geq 1\}$ be a sequence of identically distributed random variables on the probability space $(\Omega, \mathbf{P}, \mathcal{A})$. Assume that each of these random variables has mean 0 and variance 1. If A is a Borel set of real numbers whose frontier, ∂A, has Lebesgue measure 0, then*

$$\mathbf{P}\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}} \in A\right) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_A e^{-t^2/2} dt,$$

as $n \to \infty$. (The frontier *of a set A in a topological space is the set of points which are limit points for both the set and its complement.)*

Now suppose that we have independent, identically distributed random variables $y_1, y_2, \ldots$ with mean 0 and variance 1. Let $S_n = \sum_{i=1}^n y_i$. For fixed n we want to define a process $X_n(t)$ such that $X_n(k/n) = n^{-\frac{1}{2}} S_k$, for $0 \leq k \leq n$. In between the fractions we interpolate linearly:

$$X_n(t) = \frac{1}{\sqrt{n}}(S_{[nt]} + (nt - [nt])y_{[nt]+1}), \qquad 0 \leq t \leq 1, \tag{1.3}$$

where $[a]$ denotes the largest integer smaller than a. By the central limit theorem we expect that the functions $X_n(t)$ will have a limiting distribution on $C[0,1]$ such that a continuous function $X(t)$ will have $X(t) = 0$ almost surely, and for $0 \leq t_1 < \cdots < t_n \leq 1$ the increments $X(t_1)$, $X(t_2) - X(t_1)$, ..., $X(t_n) - X(t_{n-1})$ will be independent and normally distributed with means 0 and variances t_1, $t_2 - t_1$, ..., $t_n - t_{n-1}$. For such a distribution the probability of the finitary event $[X(t_j) \in A_j$ for $1 \leq j \leq n]$ would be given by Equation (1.2), where $t_0 = y_0 = 0$ and $A_1, \ldots, A_n$ are Borel subsets of $\mathbb{R}$. This would uniquely determine the limiting distribution. These remarks are made precise by Donsker's invariance principle:

Theorem 1.2. Donsker [8] *There is a probability measure W on Σ, the Borel-algebra of $C[0,1]$, such that, for a Borel subset A with $W(\partial A) = 0$, we have*

$$\mathbf{P}(X_n \in A) \to W(A)$$

as $n \to \infty$, where the functions X_n are the functions defined above by (1.3). Also, for $X \in C[0,1]$, almost surely $X(0) = 0$, and for $0 \leq t_1 < \cdots < t_n \leq 1$ the events $X(t_1)$, $X(t_2) - X(t_1)$, ..., $X(t_n) - X(t_{n-1})$ are independent and have normal distributions all of mean 0 and variances t_1, $t_2 - t_1$, ..., $t_n - t_{n-1}$, respectively.

The density function

$$g(t_1, \ldots, t_n, x_1, \ldots, x_n) = \prod_{k=1}^{n} [2\pi(t_k - t_{k-1})]^{-1/2} \exp\left[\frac{-(x_k - x_{k-1})^2}{2(t_k - t_{k-1})}\right],$$

where $t_0 = x_0 = 0$, is called the Gauss kernel. The measure in Donsker's theorem of which existence is guaranteed is known as *Wiener measure*. For Borel subsets $A_1, \ldots, A_n$ of $\mathbb{R}$ and for $0 \le t_1 < \cdots < t_n \le 1$, the probability

$$W(\{X \in C[0,1] : (X(t_1), \ldots, X(t_n)) \in A_1 \times \cdots \times A_n\})$$

is given by (1.2).

Note that we use $[0,1]$ as our time line purely for convenience; Brownian motion can be defined for any interval or for all of $\mathbb{R}$ (for each ω).

Brownian motion in n dimensions on a probability space $(\Omega, \mathbf{B}, \mathbf{P})$ is defined as the process

$$X = (X_1, \ldots, X_n) : [0,1] \times \Omega \to \mathbb{R}^n,$$

where the X_i are mutually independant one-dimensional Brownian motions.

It is sometimes useful to know how new Brownian motions can be obtained from old; the following will be used in Chapter 6.

Proposition 1.3. *Let $\{X(t) : t \in [0,1]\}$ be a Brownian motion as defined above. For fixed real numbers $s > 0$ and $\lambda \neq 0$, the following process is also a Brownian motion:*

$$\{\lambda^{-1} X(\lambda^2 t) : t \in [0,1]\}.$$

Also, $\{X(t) : t \in [0,1]\}$ and $\{tX(1/t) : t \in [0,1]\}$ have the same distribution.

Another fundamental property of Brownian motion we will depend on heavily later is the Markov property. Suppose (for the moment) that we are working with a Brownian motion X on $[0,\infty)$ instead of $[0,1]$ as usual. In its weaker form, the Markov property asserts that Brownian motion can be seen as "starting over" at each $t \in \mathbb{R}$. The future of the path in a sense just depends on the present, not on the past. Specifically, if the probability measure associated with the process is denoted by $\mathbf{P}$ (as in the definition) and $s \in \mathbb{R}$, there exists a probability $\mathbf{P}_s$ such that the process starting at time s has the same distribution as the original process; i.e.

$$\mathbf{P}\{\omega : X(\cdot) \in A\} = \mathbf{P}_s\{\omega : X(\cdot + s) \in A\},$$

where A is a Borel subset of $\mathbb{R}$. Furthermore, the process $X(t+s) - X(s)$ has the same distribution as $X(t)$. The *strong* Markov property states that Brownian motion also starts over at Markov times. A Markov time for Brownian motion is a measurable function $\sigma(\omega)$ on Ω with values in the positive reals satisfying

$$\{\omega : \sigma(\omega) < t\} \in \mathcal{B}_t$$

for all t, where $\mathcal{B}_t$ is the σ-algebra generated by $\{X(s) : s < t\}$. The strong Markov property asserts that for a Markov time $\tau(\omega)$, the process

$$X(\tau(\omega) + t) - X(\tau(\omega)), \quad 0 \leq t \leq 1$$

is a Brownian motion.

We will mostly be focusing on the *sample paths* (or *trajectories*) of Brownian motion, that is, the maps $X(\cdot)$ from $[0, 1]$ to $\mathbb{R}$ for each ω. We now go on to look at a few sample path properties, some of which will be used in the sequel.

Proposition 1.4. *Almost all sample paths of a Brownian motion are nowhere differentiable.*

The original proof of this is due to Dvoretzky, Erdős and Kakutani [9]. Although this result has no immediate bearing on this thesis, it does serve as an indication of the interesting fractal properties Brownian paths may have. A more complete introduction to such matters is given in Chapter 3; for now it suffices to mention that some of the earliest exceptional sets (what are now called fractals) occurred in the construction of functions that are nowhere differentiable. Later we shall be considering the structure of the sets of *rapid points* of a Brownian motion X. Given $0 < \alpha < 1$, these are the elements t of [0,1] for which

$$\limsup_{h\to 0} \frac{|X(t+h) - X(t)|}{\sqrt{2|h| \log 1/|h|}} \geq \alpha. \tag{1.4}$$

These sets have Lebesgue measure 0 almost surely and have rather unusual properties. They are exceptional points of rapid growth, since the usual local growth behaviour is described by Khintchine's law of the iterated logarithm [25]:

$$\mathbf{P}\left\{\limsup_{h\to 0} \frac{|X(t_0+h) - X(t_0)|}{\sqrt{2|h| \log\log 1/|h|}} = 1\right\} = 1,$$

for any prescribed $t_0 \in [0, 1]$. The modulus of continuity of a continuous function f is given by

$$\omega_f(h) = \sup_{|t_2 - t_1| \leq h} |f(t_1) - f(t_2)|.$$

For Brownian motion we find that, for small enough h,

$$\omega(h) \leq \sqrt{2C|h| \log |h|^{-1}},$$

for some constant $C > 0$. If we set

$$S(h, a, b) = \sup_{a \leq t \leq b} |X(t+h) - X(t)|,$$

then for any $0 \leq b < a$ we have, almost surely, that

$$\limsup_{h\to 0} \frac{S(h, a, b)}{\sqrt{2|h| \log 1/|h|}} = 1.$$

This result was established by Lévy [19]. With some changes in Lèvy's proof, the result can be strengthened such that the following holds, almost surely [31]:

$$\lim_{h\to 0} \frac{S(h,a,b)}{\sqrt{2|h|\log 1/|h|}} = 1.$$

Our study will be concerned primarily with the so-called fractal properties associated with these paths, that is, the Hausdorff dimensional properties. Toward the end we will also consider so-called Fourier dimensional properties. In the following section we give a brief introduction to Hausdorff dimension.

1.2 Hausdorff dimension

Before we introduce Hausdorff dimension, it might be worthwhile to briefly discuss the notion of topological dimension. Although the topological dimension is not much used in the sequel, we present a short discussion of it here in order to contrast it with the subtler Hausdorff dimension, and to hopefully provide some indication of why (and how) the second concept was a consequence of the shortcomings of the first.

The intuition behind topological dimension is an old one and can be traced back to Euclid, although his notions were somewhat imprecise. Also, the very name presupposes the existence of topology (in which field Hausdorff accomplished his most famous work [17]).Although it is easily proved that the dimensions coincide for certain sets, it is not so obvious how the two dimensions are linked in any intuitive way. It does however seem likely that the precise formulation of topological dimension (as given below), which shifts attention to the idea of topological cover, may have led to the consideration of the "size" of the cover, which leads us naturally to Hausdorff dimension. The notion had to be separately formulated, since the size of the cover is of no interest in the case of integer dimensions.

The main players in the story of topological dimension were Brouwer, Lebesgue, Menger and Urysohn. An important rôle was certainly also played by Cantor. In showing that the line and the plane have a one-to-one correspondence (*Je le vois, mais je ne le crois pas* - "I see it but I don't believe it" [28]), Cantor put to rest the notion that n-dimensionality is the same as saying that a set can be described by n parameters. In what is in retrospect a prelude to the subtleties of Brownian motion to follow, he also constructed a function that is continuous and non-constant but has a derivative 0 except on a set of Lebesgue measure 0—a so-called *singular function*, known as the "devil's staircase". A fuller account of the fascinating history of the concept of dimension is given in [28].

Formally, we can define the dimension of a set as follows [28]: The *topological dimension* (defined in a way also now known as the "covering dimension") of a compact metric space F is $\leq r$ iff for every $\varepsilon > 0$, there is a cover of order $\leq r+1$ of F by finitely many closed sets with diameter $< \varepsilon$. A cover has *order* $\leq r+1$ iff every $r+2$ distinct sets in the cover has empty intersection.

It is not hard to apply this definition in simple cases; given a closed interval of $\mathbb{R}$, any finite cover by intervals with diameter $< \varepsilon$ can reduce to a "better", or more efficient, cover in which none of the covering intervals will intersect more than 2 of the others.

We now turn to Hausdorff measure and dimension. In the definition that follows, note that if we only allowed for exponents of integer value, we would obtain something not far removed, and not richer than, topological dimension. Allowing for real numbers as exponents allows Hausdorff measure and dimension to assign non-zero values to sets which would have had a Lebesgue measure and a topological dimension of zero.

Given a compact set A on the unit interval (or any bounded subset of $\mathbb{R}$) and $\epsilon > 0$, we consider all coverings of the set by open balls B_n of diameter smaller than or equal to ϵ. For each cover we form the sum

$$\sum_{n=0}^{\infty} |B_n|^{\alpha},$$

where $|\cdot|$ denotes the diameter of a set (i.e., the maximum distance between any two points). We will call these the α-Hausdorff sums for A, always with reference to a given cover. For each A we can take the infimum over all such sums, as $\{B_n\}$ ranges over all possible covers of A:

$$S_{\alpha}^{\epsilon}(B) = \inf_{\{B_n\}} \sum_{n} |B_n|^{\alpha}.$$

As ϵ decreases to 0, $S_{\alpha}^{\epsilon}(B)$ increases to a limit $H_{\alpha}(A)$ (which might be infinite) which is called the α-Hausdorff measure of A, or the Hausdorff measure of A in dimension α (we will refer to this as just "the measure" when the context is clear). Since H_{α} is σ-subadditive but otherwise satisfies the requirements of a measure, it is an outer measure.

Definition 1.2. *The* Hausdorff dimension*, $dimA$, of a compact set $A \subseteq [0,1]$ is the supremum of all the $\alpha \in [0,1]$ for which, for any cover B of A, $S_{\alpha}(B) = \infty$. This is equal to the infimum of all $\beta \in [0,1]$ for which there exists a cover C of A such that $S_{\alpha}(B) = 0$.*

To see that the supremum of the one set of values is indeed equal to the infimum of the other, let $0 < \alpha < \beta \leq 1$ and consider the following:

$$\sum_{n} |B_n|^{\beta} \leq \sup_{n} |B_n|^{\beta-\alpha} \sum_{n} |B_n|^{\alpha}.$$

Hence, if $S_{\alpha}(A) < \infty$, $S_{\beta}(A) = 0$, or equivalently, $S_{\alpha}(A) = \infty$ if $S_{\beta}(A) > 0$.

We can now also see why the size of the cover is of no interest for integer dimensions; the α-Hausdorff sums simply diverge or are 0 on any non-integer real, as can be easily seen by dividing the unite interval into n pieces, and considering the sums $\sum n^{-\alpha}$ as $n \to \infty$.

Usually, little is known about the value of the measure H_{α} where $\alpha = \mathrm{dim}A$.

It might be valuable at this point just to give some motivation behind the creation and use of Hausdorff dimension. From Hausdorff's original paper [16] we may infer that his original intention was somewhat akin to some of the motivation behind the creation of non-standard analysis (which we shall soon be using in this context). In this paper he states:

> In this way,the dimension becomes a sort of characteristic measure of graduality similar to the 'order' of convergence to zero, the 'strength' of convergence, and related concepts.

When we formulate the basic concepts of nonstandard analysis, we shall see that it too discriminates between different rates of convergence. Hausdorff amusingly called the concept a "small contribution" to the measure theory of Lebesgue and Caratheodory. His concern seemed to be that the usual Lebesgue measure would simply assign a value of 0 to many perfect linear sets which exhibit interesting behaviour, thus effectively preventing the study of its structure according to usual measure theory. His dimension does go some way towards giving an indication of the structure of the set being studied, thus yielding far more information than Lebesgue measure (or any measure absolutely continuous thereto) would. The different measures do coincide sometimes. For instance, the Hausdorff dimension of the interval $[0, 1]$ is 1, and the corresponding measure H_1 is also 1. It should not be hard to guess what the Hausdorff dimension and corresponding measure for the empty set is, seen as a subset of $[0, 1]$. The Hausdorff dimension of the rational numbers in $[0, 1]$, or indeed of any countable set, is 0. This can be seen by ordering the set and, for any given $\alpha \in (0, 1]$, covering the kth point by an open ball of diameter $1/2^{\frac{k}{\alpha}}$. Thus, for any α, there is a cover of which the α-Hausdorff sum is 1. The supremum of the set for which a cover yields an infinite sum is therefore 0.

Although we work almost exclusively with compact sets in one (topological) dimension, it is possible to do so in any number of dimensions. The principles remain exactly the same and the Hausdorff dimension of a set is the same whether we consider it as a subset of $\mathbb{R}$ or $\mathbb{R}^n$.

An interesting class of sets one often encounters in studying Hausdorff dimension is the so-called Cantor-like sets, the most famous of which is the triadic Cantor set. These will be explained more fully in Chapter 3. For now we briefly mention some properties of Brownian paths. Hopefully this will furnish some motivation as to why these are fascinating objects of study. Nonstandard proofs of some of these will be offered in Chapter 4.

1.3 Some fractal properties of Brownian paths

A proper study of Brownian motion and Hausdorff dimension would fill far more than this introduction. It is fascinating how Brownian motion seems to yield sets with interesting dimensional properties around every corner, and also to see what its effect is on already interesting sets. We will just mention some of the more relevant "highlights" in this section. For a more complete introduction the reader is referred to Kahane's book [21].

For a Brownian path $X(t)$ and a given $a \in \mathbb{R}$ the set $\{t \in [0,1] : X(t) = a\}$ is called the *level set* of X associated to a. It is well known that these sets have a Hausdorff dimension of 1/2, almost surely, as can be seen on pp. 250-255 of Kahane's book [21]. The images of subsets of $[0,1]$ with dimensions different from 1/2 have more interesting properties: If such a set has dimension $\alpha < 1/2$, its image has dimension 2α with probability 1, and if it has dimension $\beta > 1/2$, its image is almost surely of dimension 1 and has positive Lebesgue measure. Furthermore, such a set will almost surely have an interior point (i.e., it has a non-void interior) [21]. It will become a little clearer why the dividing line is 1/2 when we consider nonstandard proofs of the above. These results can be generalised to n dimensions by replacing 1/2 by $n/2$. We shall attempt to understand some of these properties in an intuitively satisfying way by using Loeb's nonstandard measure theory. An introduction to the basics makes up Chapter 2, after which we will define Hausdorff dimension in a hyperfinite context. The remaining chapters will study first Hausdorff and then the Fourier dimensional properties, specifically of the rapid points of Brownian motion. The fourth chapter reproves some widely known results, but in what is hopefully a clearer and more "hands-on" way. In the fifth chapter we give a nonstandard proof of a theorem by Orey and Taylor [31]. What is somewhat surprising about the proof is that certain probabilities are reflected almost surely *within the paths themselves*; when we consider ratios of intervals chosen (with a certain property) to the total number of intervals, these are almost surely similar to the probability that an subinterval will have the required property. The sixth chapter is a study of Kaufman's proof that the rapid points give rise to so-called Salem sets. There is some departure from the original, since much of it is very sketchy and requires further clarification, a task which was undertaken in the final chapter of this thesis.

2 Introduction to nonstandard analysis and Loeb measure theory

Before defining Loeb measures, we briefly introduce the nonstandard universe in which we will be working. This exposition is largely based on the very clear monograph of Cutland [7]. Although Loeb measures are standard measures, their construction involves nonstandard analysis (NSA).

2.1 The hyperreals

We construct a real line ${}^*\mathbb{R}$ which is richer than the standard reals $\mathbb{R}$. This is an ordered field which extends the real numbers in two notable ways:

(i) ${}^*\mathbb{R}$ contains non-zero *infinitesimals*; that is, numbers of which the absolute values are smaller than any real number; and

(ii) ${}^*\mathbb{R}$ contains positive and negative *infinite numbers*; that is, numbers which in absolute value are larger than any real number.

We say that $x, y \in {}^*\mathbb{R}$ are infinitely close whenever $x - y$ is infinitesimal and denote it by $x \approx y$. Thus, $x \approx y$ if for every $\varepsilon > 0$ in $\mathbb{R}$, $|x - y| < \varepsilon$. The set of all such y that are infinitesimally close to x is called the *monad* of x.

There are several ways of constructing the extended universe. We shall use an ultrapower construction. An axiomatic approach is also possible, as developed by E. Nelson; see for instance [30]. We prefer to use the ultrapower construction because it is pertinent to later constructions.

Definition 2.1. *A free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ that is closed under finite intersections and supersets, contains no finite sets and for every $A \subseteq \mathbb{N}$ has either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$.*

Given such a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ we construct ${}^*\mathbb{R}$ as an ultrapower of the reals

$${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathcal{U}.$$

The set ${}^*\mathbb{R}$ that we obtain therefore consists of equivalence classes of sequences of reals under the equivalence relation $\equiv_{\mathcal{U}}$, where

$$(a_n) \equiv_{\mathcal{U}} (b_n) \Leftrightarrow \{n : a_n = b_n\} \in \mathcal{U}.$$

The equivalence class of a sequence (a_n) is denoted by either $(a_n)_{\mathcal{U}}$ or, in the sequel, by $\langle a_n \rangle_{\mathcal{U}}$. It is clear that ${}^*\mathbb{R}$ is then an extension of $\mathbb{R}$, the usual real numbers represented by equivalence classes of constant sequences. The usual algebraic operations such as $+, \times, <$ are easily extended, but shall be denoted in the usual way. Functions and relations on $\mathbb{R}$ can be extended pointwise without difficulty. Exactly which properties of ${}^*\mathbb{R}$ are inherited from $\mathbb{R}$ is specified in the following theorem, a restricted version of the more general *transfer principle*:

Theorem 2.1. *Let φ be any first order statement. Then φ holds in $\mathbb{R}$ if and only if ${}^*\varphi$ holds in ${}^*\mathbb{R}$.*

A first order statement φ (or ${}^*\varphi$ in ${}^*\mathbb{R}$) is one referring to elements (fixed or variable) of $\mathbb{R}$ (respectively, ${}^*\mathbb{R}$) and to fixed functions and relations on $\mathbb{R}$ (respectively, ${}^*\mathbb{R}$), that uses the usual logical connectives *and* ($\wedge$), *or* ($\vee$), *implies* ($\rightarrow$) and *not* ($\neg$). Quantification may be done over elements but not over relations or functions; that is, $\forall x$, $\exists y$ are allowed, but $\forall f$, $\exists R$ are not. As an example, the density of the rationals in the reals can be written as

$$\forall x \in \mathbb{R} \forall y \in \mathbb{R}(x < y \rightarrow \exists z \in \mathbb{R}(z \in \mathbb{Q} \wedge (x < z < y))),$$

an expression meaning, "between every two reals is a rational". From the transfer principle we can therefore immediately conclude that the statement is true in ${}^*\mathbb{R}$, that is, that the hyperrationals are dense in the hyperreals. The corresponding transferred statement is as follows:

$$\forall X \in {}^*\mathbb{R} \forall Y \in {}^*\mathbb{R}(X < Y \rightarrow \exists Z \in {}^*\mathbb{R}(Z \in {}^*\mathbb{Q} \wedge (X < Z < Y))),$$

In transferring, every set, relation and function in the original statement is replaced by its nonstandard extension, according to the ultrapower construction.

We say that an element x of ${}^*\mathbb{R}$ is finite if there is some $r \in \mathbb{R}$ such that $x < |{}^*r|$. A simple but important theorem is the following:

Theorem 2.2. *If $x \in {}^*\mathbb{R}$ is finite, then there is a unique $r \in \mathbb{R}$ such that $x \approx r$. Any finite hyperreal is thus expressible as $x = r + \delta$ with $r \in \mathbb{R}$ and δ infinitesimal.*

Proof. [1]Suppose $x \in {}^*\mathbb{R}$ is finite. Let D_1 be the set of $r \in \mathbb{R}$ such that ${}^*r < x$ and D_2 the set of $r' \in \mathbb{R}$ such that $x < {}^*r'$. The pair (D_1, D_2) forms a Dedekind cut in $\mathbb{R}$, hence determines a unique $r_0 \in \mathbb{R}$. A simple argument shows that $|x - {}^*r_0|$ is infinitesimal. $\square$

We call r_0 in the above theorem the *standard part* of x and denote it as either ${}^\circ x$ or as $\mathrm{st}(x)$. Both are used, sometimes in conjunction, to improve readability.

The following theorem will find application in the next chapter.

Theorem 2.3. *Let (s_n) be a sequence of real numbers and let $l \in \mathbb{R}$. Then*

$$s_n \rightarrow l \text{ as } n \rightarrow \infty \Longleftrightarrow {}^*s_K \approx l \text{ for all infinite } K \in {}^*\mathbb{N}.$$

Proof. [7] Suppose that $s_n \rightarrow l$ and let $K \in {}^*\mathbb{N}$ be a fixed infinite number. We must show, for all real $\varepsilon > 0$, that $|{}^*s_K - l| < \varepsilon$. From ordinary real analysis we know that there exists some $n_0 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N} \, [n \geq n_0 \rightarrow |s_n - l| < \varepsilon].$$

According to the transfer principle, the following is true in ${}^*\mathbb{R}$:

$$\forall N \in {}^*\mathbb{N} \, [N \geq n_0 \rightarrow |{}^*s_N - l| \leq \varepsilon].$$

In particular, $|{}^*s_K - l| < \varepsilon$ as required.
Conversely, suppose that ${}^*s_K \approx l$ for all infinite $K \in {}^*\mathbb{N}$. For any given real $\varepsilon > 0$ we have

$$\exists K \in {}^*\mathbb{N}\, \forall N \in {}^*\mathbb{N}\, [N \geq K \rightarrow |{}^*s_N - l| < \varepsilon].$$

By transferring this "down" to $\mathbb{R}$, we get

$$\exists k \in \mathbb{N}\ \ \forall n \in \mathbb{N}\, [n \geq k \rightarrow |s_n - l| < \varepsilon].$$

By then taking n_0 any of such extant k, we have that $s_n \rightarrow l$. $\square$

2.2 The nonstandard universe

The principles of the previous section can be used in a much broader context than just real analysis. Given any mathematical object $\mathcal{M}$ (whether it is a group, ring, vector space, etc.), we can construct a nonstandard version ${}^*\mathcal{M}$. We use a somewhat more economical construction however, by starting with a working portion of the mathematical universe $\mathbb{V}$ and ending up with a ${}^*\mathbb{V}$ which will contain ${}^*\mathcal{M}$ for every $\mathcal{M} \in \mathbb{V}$. This has the added advantage of preserving some of the relations between structures through the more general transfer principle.

We start with the superstructure over $\mathbb{R}$, denoted by $\mathbb{V} = V(\mathbb{R})$. It is defined as follows:

$$\begin{aligned} V_0(\mathbb{R}) &= \mathbb{R} \\ V_{n+1}(\mathbb{R}) &= V_n(R) \cup \mathcal{P}(V_n(\mathbb{R})), \quad n \in \mathbb{N} \\ \mathbb{V} &= \bigcup_{n \in \mathbb{N}} V_n(\mathbb{R}). \end{aligned}$$

($\mathcal{P}(A)$ denotes the power set of the set A.)

If a larger (or simply different) universe is required, start the same process with a more suitable set than $\mathbb{R}$.

Next one must construct a mapping $^* : V(\mathbb{R}) \rightarrow V({}^*\mathbb{R})$ associating to an $\mathcal{M} \in \mathbb{V}$ a nonstandard extension ${}^*\mathcal{M} \in V({}^*\mathbb{R})$. The nonstandard universe can now be constructed by means of an ultrapower

$$\mathbb{V}^{\mathbb{N}}/\mathcal{U},$$

and then utilising a "Mostowski collapse" [1]. This is somewhat more complicated to do than in the case of ${}^*\mathbb{R}$ and we do not go into detail here. It is sufficient to consider the nonstandard universe as the set of objects

$${}^*\mathbb{V} = \{x : x \in {}^*\mathcal{M} \text{ for some } \mathcal{M} \in \mathbb{V}\}.$$

Sets in ${}^*\mathbb{V}$ are called *internal* sets. It should be noted that ${}^*\mathbb{V} \in V({}^*\mathbb{R})$, but that $V({}^*\mathbb{R})$ contains sets that are not internal.

We now also have a transfer principle which specifies which statements may be moved from one structure to the other. A *bounded quantifier statement* is a statement which can be written so that all quantifiers range over a fixed set. Thus, quantifiers like $\forall x \in A$ or $\exists y \in B$ are allowed, but not unbounded quantifiers such as $\forall x$ and $\exists y$. Note that often boundedness is implied in the exposition and is not always specifically indicated in the statement. When given a bounded quantifier statement φ, we obtain its nonstandard version ${}^*\varphi$ by replacing every set, function or relation in φ by its nonstandard counterpart. Specifically, since we can consider relations and functions as sets as well, we replace each set A by its nonstandard counterpart ${}^*A$, whilst the logical connectives in the statement φ remain the same. Thus, a variable x ranging over $\mathbb{R}$ becomes a variable X ranging over ${}^*\mathbb{R}$, a function f is replaced by its extension ${}^*f$, and a relation R is replaced by its extension ${}^*R$.

Theorem 2.4. *A bounded quantifier statement φ holds in $\mathbb{V}$ if and only if ${}^*\varphi$ holds in ${}^*\mathbb{V}$.*

The transfer principle can after some consideration be seen to apply only to *internal* sets. For instance, the concept of supremum implies that each bounded set will have a least upper bound. However, $\mathbb{N}$ seen as a member of ${}^*\mathbb{R}$ is bounded, but has no supremum. It is therefore an *external* (i.e. non-internal) set.

We show now that the concept of supremum transfers. The proof also provides an illustration of how to change a bounded quantifier statement φ into ${}^*\varphi$.

Proposition 2.5. *Every nonempty internal subset of ${}^*\mathbb{R}$ with an upper bound has a least upper bound.*

Proof. The notation used in this proof refers back to our construction of the nonstandard universe. We express the fact that any nonempty subset of the standard real numbers has a least upper bound by the statement

$$\begin{aligned}\Phi(\mathbb{R}, V_2(\mathbb{R})) = \forall A \in V_2(\mathbb{R})[A \neq \emptyset \wedge (\exists x \in \mathbb{R}(\forall y \in A(y < x))) \to \\ \exists z \in \mathbb{R}(\forall y \in A(y < x) \wedge \forall u \in \mathbb{R}\forall y \in A\,(y \leq u \to z \leq u))].\end{aligned}$$

Since the statement $\Phi = \Phi(\mathbb{R}, V_2(\mathbb{R}))$ is true in $V(\mathbb{R})$, the transferred ${}^*\Phi = \Phi({}^*\mathbb{R}, {}^*V_2(\mathbb{R}))$ condition is true in $V({}^*\mathbb{R})$. The nonstandard version of the above statement that will hold is

$$\begin{aligned}\Phi({}^*\mathbb{R}, {}^*V_2(\mathbb{R})) = \forall A \in {}^*V_2(\mathbb{R})[A \neq \emptyset \wedge (\exists X \in {}^*\mathbb{R}\,(\forall Y \in A(Y < X))) \to \\ \exists Z \in {}^*\mathbb{R}\,(\forall Y \in A(Y < X) \wedge \forall u \in {}^*\mathbb{R}\,\forall Y \in A\,(Y \leq U \to Z \leq U))].\end{aligned}$$

(The capitals for the variables are not necessary and just serve to indicate that the statement is indeed nonstandard.) □

The transfer principle yields the following properties, which will be used later:

Proposition 2.6. *Let $A \subseteq {}^*\mathbb{R}$ be an internal set.*

(i) If A contains arbitrarily large finite numbers, then it also contains an infinite number.

(ii) If A contains arbitrarily small positive infinite numbers, then it contains a positive finite number.

These two are known as the *overflow* and *underflow* properties, respectively. We give the proof as another illustration of the use of the Transfer Principle.

Proof.

(i) Since A is an internal set, if it has an upper bound, it must have a least upper bound. However, if it did not contain an infinite number, it would be bounded by any infinite number. Such a number would necessarily be infinite, leading to a contradiction.

(ii) The same type of proof as in (i) holds here, once it is recognised that the transfer principle guarantees that an internal set bounded from below has an infimum. □

Note that these properties are also easily obtained from the ultrafilter construction. By taking reciprocals, similar properties can be seen to hold for infinitesimals.

An important property of any nonstandard universe constructed as an ultrapower is that of $\aleph_1$-saturation:

Proposition 2.7. *If $(A_m)_{m\in\mathbb{N}}$ is a countable decreasing sequence of nonempty internal sets, then $\cap_{m\in\mathbb{N}} A_m \neq \emptyset$.*

A useful reformulation of this is known as *countable comprehension*: Given any sequence $(A_n)_{n\in\mathbb{N}}$ of internal subsets of an internal set A, there is an *internal* sequence $(A_n)_{n\in{}^*\mathbb{N}}$ of subsets of A that extends the original sequence. This property will be used in the construction of Loeb measure.

2.3 Nonstandard topology

Before doing analysis in a nonstandard universe, we must clearly understand the topology. Firstly, we see that the concept of being infinitely close, and therefore the idea of a *monad*, can be extended:

Definition 2.2. *Let (X, τ) be a topological space.*

(i) For $a \in X$ the monad of a is

$$monad(a) = \bigcap_{a\in U\in\tau} {}^*U.$$

(ii) For $x \in {}^*X$, we write $x \approx a$ if $x \in monad(a)$.

(iii) $x \in^* X$ *is said to be nearstandard if* $x \approx a$ *for some* $a \in X$.

(iv) *For any* $Y \subseteq {}^*X$, *we denote the nearstandard points in* Y *by* $ns(Y)$.

(v) $st(Y) = \{a \in X : x \approx a \text{ for some } x \in Y\}$ *is called the standard part of* Y.

The following result allows us to generalise the pointwise standard part mapping:

Proposition 2.8. *A topological space* X *is Hausdorff if and only if*

$$monad(a) \cap monad(b) = \emptyset \text{ for } a \neq b, \quad a, b \in X.$$

This means we can define the function

$$\text{st} : \text{ns}({}^*X) \to X$$

as

$$\text{st}(x) = \text{ the unique } a \in X \text{ with } a \approx x.$$

Again, we use the notation ${}^\circ x = \text{st}(x)$ interchangeably.
We mention some general topological results.

Proposition 2.9. *Let* (X, τ) *be separable and Hausdorff. Suppose* $Y \subseteq {}^*X$ *is internal and* $A \subseteq X$. *Then*

(i) $st(Y)$ *is closed,*

(ii) *if* X *is regular and* $Y \subseteq ns({}^*X)$, *then* $st(Y)$ *is compact,*

(iii) $st({}^*A) = \overline{A}$ *(closure of* A*),*

(iv) *if* X *is regular, then* A *is relatively compact iff* ${}^*A \subseteq ns({}^*X)$.

Since we will be dealing almost exclusively with continuous functions, we should introduce corresponding notions in the nonstandard universe.

Definition 2.3. *Let* Y *be a subset of* ${}^*X$ *for some topological space* X *and let* $F : {}^*X \to {}^*\mathbb{R}$ *be an internal function. Then* F *is said to be* S-continuous on Y *if for all* $x, y \in Y$ *we have*

$$x \approx y \Rightarrow F(x) \approx F(y).$$

The following result allows us to switch from the one notion of continuity to another.

Theorem 2.10. *If* $F : {}^*X \to {}^*X$ *is* S*-continuous on an interval* ${}^*[a, b]$ *for real* a, b *and* $F(x)$ *is finite for some* $x \in {}^*[a, b]$, *then the standard function defined in* $[a, b]$ *by*

$$f(t) =^\circ F(t)$$

is continuous and ${}^*f(\tau) \approx F(\tau)$ *for all* $\tau \in {}^*[a, b]$.

Given a real function f defined on an interval $[a, b]$, we shall call any function F on ${}^*[a, b]$ such that $f(t) =^\circ F(t)$, a *lifting* of f.

2.4 Loeb measure

A Loeb measure is a standard measure, but constructed from a nonstandard one. That is, the Loeb measure exists on a σ-algebra and obeys all the usual rules for a measure, for example countable additivity.

We start with a given internal set Ω, an algebra $\mathcal{A}$ of internal subsets of Ω and a finite internal finitely additive measure μ on $\mathcal{A}$. Thus μ is a function from $\mathcal{A}$ to ${}^*[0,\infty)$ such that $\mu(\Omega) < \infty$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ for disjoint $A, B \in \mathcal{A}$. (We focus only on bounded Loeb measures; infinite ones shall not concern us in the sequel.) We can then define the mapping

$$^\circ\mu : \mathcal{A} \to [0,\infty)$$

by $^\circ\mu(A) = ^\circ(\mu(A))$. This is finitely additive and therefore $(\Omega, \mathcal{A}, \mu)$ is a standard finitely additive measure space. This is not usually a measure, since $^\circ\mu$ is not always σ-additive. We shall see shortly, however, that it is *almost* a measure. The following crucial theorem was proved by Loeb [27]. It is possible to give a quick proof using Caratheodory's extension theorem, but we shall follow Cutland [7] and give a more straightforward approach.

Theorem 2.11. *There is a unique σ-additive extension of $^\circ\mu$ to the σ-algebra $\sigma(A)$ generated by $\mathcal{A}$. The measure theoretic completion of this measure is the Loeb measure associated with μ, denoted by μ_L. The completion of $\sigma(A)$ is the Loeb σ-algebra, denoted by $L(\mathcal{A})$.*

The more straightforward proof depends on the notion of a Loeb null set:

Definition 2.4. *Let $B \subseteq \Omega$, where B is not necessarily internal. We call B a* Loeb null set *if for each standard real $\varepsilon > 0$ there is a set $A \in \mathcal{A}$ with $B \subseteq A$ and $\mu(A) < \varepsilon$.*

This allows us to make precise the notion that $\mathcal{A}$ is *almost* a σ-algebra:

Lemma 2.12. *Let $(A_n)_{n\in\mathbb{N}}$ be an increasing family of sets, with each A_n in $\mathcal{A}$ and let $B = \bigcup_{n\in\mathbb{N}} A_n$. Then there is a set $A \in \mathcal{A}$ such that*

(i) $B \subseteq A$

(ii) $^\circ\mu(A) = \lim_{n\to\infty} {}^\circ\mu(A_n)$ *and*

(iii) $A \setminus B$ *is null*

Proof. Let $\alpha = \lim_{n\to\infty} {}^\circ\mu(A_n)$. For any finite n,

$$\mu(A_n) \leq {}^\circ\mu(A_n) + \frac{1}{n} \leq \alpha + \frac{1}{n}.$$

Let $(A_N)_{N\in{}^*\mathbb{N}}$ be a sequence of sets in $\mathcal{A}$ extending the sequence $(A_n)_{n\in\mathbb{N}}$, made possible by $\aleph_1$ saturation (see 2.2). The overflow principle then guarantees an infinite N such that

$$\mu(A_N) \leq \alpha + \frac{1}{N}.$$

If we now let $A = A_N$, (i) will hold because $A_n \subseteq A$ for each n. Also, $\mu(A_n) \leq \mu(A)$ for finite n, so ${}^\circ\mu(A_n) \leq^\circ \mu(A) \leq \alpha$ and therefore ${}^\circ\mu(A) = \alpha$. This gives (ii). For (iii), note that $A \setminus B \subseteq A \setminus A_n$ and ${}^\circ\mu(A \setminus A_n) = {}^\circ\mu(A) - {}^\circ\mu(A_n) \to 0$ as $n \to \infty$. □

Thus $\mathcal{A}$ is a σ-algebra modulo null sets. We can now define the concepts *Loeb measurable* and *Loeb measure* exactly:

Definition 2.5. *(i) Let $B \subseteq \Omega$. We say that B is* Loeb measurable *if there is a set $A \in \mathcal{A}$ such that $A \triangle B$ (the symmetric difference of A and B) is Loeb null. The collection of all the Loeb measurable sets is denoted by $L(\mathcal{A})$. The algebra $L(\mathcal{A})$ is known as the* Loeb algebra.

(ii) For $B \in L(\mathcal{A})$ define

$$\mu_L(B) = {}^\circ\mu(A)$$

for any $A \in \mathcal{A}$ with $A \triangle B$ null. We call $\mu_L(B)$ the Loeb measure *of B.*

This brings us to the central theorem of Loeb measure theory.

Theorem 2.13. *$L(\mathcal{A})$ is a σ-algebra and μ_L is a complete σ-additive measure on $L(A)$.*

The measure space $\Omega = (\Omega, L(\mathcal{A}), \mu_L)$ is called the Loeb space given by $(\Omega, \mathcal{A}, \mu)$. If $\mu(\Omega) = 1$, we refer to Ω as a *Loeb probability space.*

2.5 Loeb counting measure

We devote a short but separate section to the idea of counting measures because the idea is prominent throughout the sequel.

Let $\Omega = \{1, 2, \ldots, N\}$, where $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. The set Ω is internal. Define the counting probability ν on Ω by

$$\nu(A) = \frac{|A|}{N},$$

for $A \in {}^*\mathcal{P}(\Omega) = \mathcal{A}$. The cardinality function $|\cdot|$ transfers, so $|A|$ can be interpreted as an extension of finite, standard cardinality. The Loeb counting measure ν_L is the completion of the extension to $\sigma(A)$ of the finitely additive measure ${}^\circ\nu$.

An easy but important example of the use of such a counting measure is in the construction of Lebesgue measure.

Definition 2.6. *Fix $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ and let $\triangle t = N^{-1}$. The hyperfinite time line for the interval $[0, 1]$ based on the infinitesimal $\triangle t$ is the set*

$$\mathbf{T} = \{0, \triangle t, 2 \triangle t, 3 \triangle t, \ldots, 1- \triangle t\}.$$

The following theorem provides an intuitive construction of Lebesgue measure.

Theorem 2.14. *Let ν_L be the Loeb counting measure on* $\mathbf{T}$. *Define*

(i) $\mathcal{M} = \{B \subseteq [0,1] : st_{\mathbf{T}}^{-1}(B)$ *is Loeb measurable*$\}$, *where* $st_{\mathbf{T}}^{-1}(B) = \{t \in \mathbf{T} :{}^{\circ} t \in B\}$.

(ii) $\lambda(B) = \nu_L(st_{\mathbf{T}}^{-1}(B))$ *for* $B \in \mathcal{M}$.

Then $\mathcal{M}$ *is the completion of the Borel sets* $\mathcal{B}[0,1]$ *and* λ *is Lebesgue measure on* $\mathcal{M}$.

Proof. We only need to sketch the proof here. A more complete proof can be found in [1]. To check that $\mathcal{M}$ is a σ-algebra is not difficult. Furthermore, $\mathcal{M}$ contains each standard interval $[a,b]$, since $\mathrm{st}_{\mathbf{T}}^{-1}([a,b]) = \cap_n \in \mathbb{N}({}^*[a-1/n, b+1/n] \cap \mathbf{T})$, a countable intersection of internal sets. It can also be shown that λ is a complete probability measure on $\mathcal{M}$. We can then show that λ is translation invariant and $\lambda([a,b]) = b-a$. The measure is therefore an extension of Lebesgue measure. Now take $B \in \mathcal{M}$ and an internal A such that $A \subseteq \mathrm{st}_{\mathbf{T}}^{-1}(B)$ is an inner approximation. The set $\mathrm{st}(A)$ is a closed inner approximation of B; enough to show that B is Lebesgue measurable. $\square$

In Chapter 4 we will encounter another important use of counting measure in constructing Wiener measure.

3 A nonstandard version of Hausdorff dimension

In this chapter we show that a formulation of Hausdorff measure as a nonstandard counting measure, similar to Loeb's formulation of Lebesgue measure, is possible and prove some well-known theorems using these nonstandard techniques. It turns out that some interesting dimensional properties of Brownian paths become quite easy to prove using hyperfinite counting arguments.

Before we start the proof, we need a nonstandard version of the following result known as Frostman's lemma [13]. We denote the β-dimensional Hausdorff measure of a set A by $\text{meas}_\beta A$.

Theorem 3.1. *(Frostman's lemma) Let A be a compact subset of $[0,1]$ and $\beta \in (0,1)$. Then $meas_\beta A > 0$ if and only if there exists a probability measure μ on A such that $\mu(B) \leq C|B|^\beta$ for each interval $B \subseteq [0,1]$ and some positive C.*

This is often used to prove Frostman's theorem, which we will include at the end of this chapter. A version of the lemma on the hyperfinite time line is as follows. Note that we abuse the notation slightly by using ${}^\circ\left(\frac{|A'|}{2^{N\alpha}}\right) > 0$ to mean either that the standard part of the expression in brackets exists and is larger than 0, or that the expression is infinite.

Theorem 3.2. *Let A be a compact subset of $[0,1]$. Suppose $\mathbf{T}$ is a hyperfinite time line based on the dyadic sequence $\{2^n\}$ and $A' \subseteq \mathbf{T}$ is such that its standard part is A. If ${}^\circ\left(\frac{|A'|}{2^{N\alpha}}\right) > 0$, there exists a nonstandard measure μ on a hyperfinite time line on $[0,1]$ such that the Loeb measure $\mu_L \in M^+(A)$ (the set of strictly positive measures on A) associated to μ has the property that for an absolute constant C and an arbitrary interval $B \subseteq [0,1]$, it is true that $\mu_L(B) \leq \|B\|^\alpha$.*

Proof. The measure in question is not quite as simple as, for instance, the counting measure we used to generate Lebesgue measure. In this case we have to take into account how "close" elements of A are to each other and a uniform counting measure cannot provide that information. Thus the construction of the measure is not generic but will depend specifically on the nature of A.

We use a time line based on the hyperfinite number 2^N, where $N = \langle 1,2,3,...\rangle_{\mathcal{U}}$. The measure is constructed in a number of stages, at each stage ensuring that the inequality $\mu_m(B) \leq \|B\|^\alpha$ holds, and then showing that the total measure of the interval is larger than 0 and normalising. On a dyadic interval B of order m, meaning that the interval has length 2^{-m}, count the number of elements of $B_\mathbf{T} \cap A_\mathbf{T}$ and distribute the mass $\|B\|^\alpha$ evenly over the elements of $B_\mathbf{T} \cap A_\mathbf{T}$. Thus each element of $A_\mathbf{T}$ in $B_\mathbf{T}$ receives a weight of

$$\frac{\|B\|^\alpha}{|B_\mathbf{T} \cap A_\mathbf{T}|}.$$

This does guarantee that the required inequality is true for this interval, but we must bear in mind that the measure must be additive. To this effect we go back one step, to dyadic intervals of order $m-1$. Suppose that the above interval B is contained in an interval B' of order $m-1$. We must now check whether

$$\mu(B') \leq \frac{2^\alpha \|B\|^\alpha}{|B'_{\mathbf{T}} \cap A_{\mathbf{T}}|}.$$

If adjacent intervals of order m both contain elements of $A_{\mathbf{T}}$, we will need to multiply the measure on each of these intervals by a factor of $\frac{2^\alpha}{2}$. We continue doing this until we cover all dyadic intervals on the time line, both standard and nonstandard. Thus the measure is finitely additive in the nonstandard context. Also, the smallest the measure of any element of $A_{\mathbf{T}}$ can be, will be

$$\frac{2^\alpha}{2} 2^{-(N-1)\alpha} = 2^{2\alpha-1} 2^{-N\alpha}.$$

Thus, the smallest the total mass over all of $A_{\mathbf{T}}$ can be is

$$2^{2\alpha-1} \frac{|A_{\mathbf{T}}|}{2^{N\alpha}}.$$

But since we have that ${}^\circ\left(\frac{|A_{\mathbf{T}}|}{2^{N\alpha}}\right) > 0$, we know there will exist some finite (but not infinitesimal) γ such that

$$2^{2\alpha-1} \frac{|A_{\mathbf{T}}|}{2^{N\alpha}} > 2^{2\alpha-1}\gamma.$$

We normalise using the total mass and obtain, for any dyadic interval B that

$$\mu(B) \leq \frac{1}{\gamma} \|B\|^\alpha.$$

The inequality will then also hold for μ_L. An arbitrary interval D will always be contained in two such dyadic intervals and therefore

$$\mu_L(D) \leq \frac{2}{\gamma} \|D\|^\alpha.$$

□

We prove the main result of this chapter in two separate theorems. The first guarantees the existence of a subset of a time line from which we can compute the dimension and the second shows that the choice of set is not very important. It is proved for subsets of $[0,1]$ only, but note that it can easily be extended to any compact interval and arbitrary (finite) dimension.

Theorem 3.3. *Given a compact subset A of $[0,1]$, there is a subset $A_{\mathbf{T}}$ on the hyperfinite time line $\mathbf{T}$ and a hyperfinite number $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that ${}^\circ A_{\mathbf{T}} = A$ and*

$$\begin{aligned} {}^\circ\left(\frac{|A_{\mathbf{T}}|}{N^\beta}\right) &= \infty \text{ for } \beta < \alpha \\ {}^\circ\left(\frac{|A_{\mathbf{T}}|}{N^\beta}\right) &= 0 \text{ for } \beta > \alpha \end{aligned}$$

if and only if $\dim A = \alpha$.

Proof. Suppose that $\beta < \dim A$. We know that the sum diverges to infinity as the sizes of the intervals decrease. Thus there will be some $N \in \mathbb{N}$ such that the β-Hausdorff sum will be larger than 1 for covers constituting of sets with diameter smaller than 2^{-K}, for all $K > N$.

In the following we will state as a bounded quantifier statement that this will hold for any cover and that such a cover always exists, a seemingly trivial point in the standard case, but not as obvious in the nonstandard. We also use the fact that we may require our intervals not to border on each other, for otherwise the Hausdorff sum may be made smaller, and we require that even the smallest sum must be larger than 1. The use of $K > J$ is justified by the compactness of the set A; that is, the cover will have a finite subcover, and therefore there will be a set of diameter no smaller than 2^{-K} for some $K > J$.

Let $S = S(A, X, K, J)$ be the following statement, where $X \subset \mathbb{N} \times \mathbb{N}$:

$$\begin{aligned} S = \quad & \forall x \in A \exists (i,j) \in X \left(x \in \left(\tfrac{i}{2^K}, \tfrac{j}{2^K}\right]\right) \wedge \forall (i,j) \in X \exists x \in A \left(x \in \left(\tfrac{i}{2^K}, \tfrac{j}{2^K}\right]\right) \\ & \wedge \forall (i,j) \in X \left(2^{-K} \leq (j-i)2^{-K} \leq 2^{-J}\right) \\ & \wedge \left[(i,j) \in X \Rightarrow \neg(\exists k \in \{0,1,\ldots,2^K-1\}((j,k) \in X))\right], \end{aligned}$$

and let $T = T(X, K, \beta)$ be the statement

$$\sum_{(i,j)\in X} \left(\frac{j-i}{2^K}\right)^\beta > 1.$$

We then express $\beta < \dim A$ as:

$$\begin{gathered} [\exists N \in \mathbb{N} \forall J > N \forall K \geq J \forall X \subseteq \{0,1,\ldots,2^K-1\} \times \{0,1,\ldots,2^K-1\} \\ (S \Rightarrow T)] \quad \wedge \\ [\exists N \in \mathbb{N} \forall J > N \forall K \geq J \exists X \subseteq \{0,1,\ldots,2^K-1\} \times \{0,1,\ldots,2^K-1\} \\ (S \Rightarrow T)]. \end{gathered}$$

The transferred statement now reads as

$$\begin{gathered} [\exists N \in {}^*\mathbb{N} \forall J > N \forall K \geq J \forall X \subseteq \{0,1,\ldots,2^K-1\} \times \{0,1,\ldots,2^K-1\} \\ ({}^*S \Rightarrow {}^*T)] \quad \wedge \\ [\exists N \in {}^*\mathbb{N} \forall J > N \forall K \geq J \exists X \subseteq \{0,1,\ldots,2^K-1\} \times \{0,1,\ldots,2^K-1\} \\ ({}^*S \Rightarrow {}^*T)], \end{gathered}$$

where ${}^*S$ and ${}^*T$ are the transferred versions of the statements S and T. Note that this necessitates replacing only A with ${}^*A$ in the original.

We now choose any sufficiently large $J \in {}^*\mathbb{N} \setminus \mathbb{N}$. The statement will still hold if we set $K = J$. This results in a "cover" of ${}^*A$ by intervals of diameter 2^{-K}. Set

$$A_{\mathbf{T}} = \left\{ \frac{j}{2^K} : (j-1, j) \in X \right\},$$

where X is the set the existence of which is guaranteed in the second line of the previous transferred statement.

By the transferred statement we know that $\sum_{(i,j)\in X}\left(\frac{j-i}{2^K}\right)^\beta > 1$, but $j-i=1$ because of the choice of K — all the infinitesimal intervals are now of the same size. Also, $|A_{\mathbf{T}}| = |X|$; therefore $\frac{|A_{\mathbf{T}}|}{2^{K\beta}} > 1$. Thus,

$$\mathrm{meas}_\beta A > 0 \Rightarrow \exists A_{\mathbf{T}} \subseteq \mathbf{T}, K \in {}^*\mathbb{N}\setminus\mathbb{N} \text{ such that } {}^\circ A_{\mathbf{T}} = A \text{ and } {}^\circ\left(\frac{|A_{\mathbf{T}}|}{2^{K\beta}}\right) > 0.$$

Since the converse holds by the nonstandard Frostman lemma, the theorem is proved.

We now show that any set which satisfies certain of the above properties is rich enough to yield Hausdorff dimension.

Theorem 3.4. *Consider a hyperfinite time line* $\mathbf{T}$ *based on the infinitesimal* 2^N*, for a given* $N \in {}^*\mathbb{N}\setminus\mathbb{N}$*. Suppose that a subset* A' *of the time line is such that* $st(A') = A$ *and for some* $\alpha > 0$

$$
\begin{aligned}
{}^\circ\left(\frac{|A'|}{2^{N\beta}}\right) &> 0 \text{ for } \beta < \alpha \text{ and} &\quad (3.1)\\
{}^\circ\left(\frac{|A'|}{2^{N\beta}}\right) &= 0 \text{ for } \beta > \alpha. &\quad (3.2)
\end{aligned}
$$

Then $\alpha = dim A$.

Proof. Given (1), the nonstandard version of Frostman's lemma immediately implies that $\dim A \geq \alpha$. For the converse inequality, notice that the second condition implies that for each $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$,

$$\frac{|A'|}{2^{N\beta}} < \varepsilon,$$

which implies the following nonstandard statement for each positive $\varepsilon \in \mathbb{R}$:

$$\exists N \in {}^*\mathbb{N}\exists Y \subseteq \{0,1,\ldots,2^N-1\}\forall x \in A'\exists i \ \in Y\left(x \in (i2^{-N},(i+1)2^{-N}]\right) \wedge \left(\tfrac{|Y|}{2^{N\beta}} < \varepsilon\right).$$

Transferring down to the standard case, we find that for each $\varepsilon > 0$,

$$\exists n \in \mathbb{N}\exists y \in \{0,1,\ldots,2^n-1\}\forall x \in {}^\circ A'\exists i \in y\,(x \in (i2^{-n},(i+1)2^{-n}]) \wedge \left(\tfrac{|y|}{2^{n\beta}} < \varepsilon\right).$$

This implies that $\mathrm{meas}_\beta A = 0$ and therefore that $\dim A \leq \alpha$. □

For computational purposes it is therefore enough to find a set in the time line with standard part A that satisfies the conditions in the above theorem. This fact will be used in subsequent chapters. In the sequel we refer to $|A_{\mathbf{T}}| \bigtriangleup t^\beta$ as nonstandard (or NS) β-Hausdorff measure and to $\mathrm{meas}_\beta A$ as just β-Hausdorff measure.

Several of the properties of the standard β-Hausdorff measure can easily be seen to be valid in the nonstandard case, such as its outer measure properties, invariance under translation (and rotation, in the multidimensional case) and homogeneity of degree β with respect to dilation.

To illustrate some applications of this formulation, we first turn to the perennial example of a set of non-integer dimension, the triadic Cantor set (construction described in any book on fractals). We did not prove that its dimension was $\alpha = \log 2/\log 3$, but do so now. The "base-infinitesimal" of the construction is $\langle 1, 3^{-1}, 3^{-2}, \ldots, 3^{-k}, \ldots\rangle_{\mathcal{U}} = \triangle t = 1/N$. The cardinality of the NS Cantor set $|A_{\mathbf{T}}|$ is given by $|A_{\mathbf{T}}|\langle 1, 2/3, 4/9, \ldots, (2/3)^k, \ldots\rangle_{\mathcal{U}} N$. The NS β-Hausdorff measure of A is then given by

$$\begin{aligned} |A_{\mathbf{T}}| \triangle t^{\beta} &= \langle (2/3)^k\rangle_{\mathcal{U}} N \langle (1/3)^{k\beta}\rangle_{\mathcal{U}} \\ &= \langle (2/3^{\beta})^k\rangle_{\mathcal{U}}, \end{aligned}$$

where we have used the obvious notation, $\langle a^k\rangle_{\mathcal{U}}$ instead of $\langle a, a^2, \ldots, a^k, \ldots\rangle_{\mathcal{U}}$. The above expression then has value 1 for $\beta = \log 2/\log 3$, which is then $\dim A$ by our previous theorems. Since the standard β-Hausdorff sum for the triadic Cantor set is also 1 for $\beta = \dim A$, we suspect that the standard parts of the nonstandard sum will be equal to the standard sum at $\dim A$ for other sets as well. This remains to be proved. We now turn to Cantor-like sets in general. In [31], a Cantor-like set K is a set generated by nested intervals in the following way:

1. Let, for each m, $I_{m,i}$ be an interval contained in $[0,1]$, where $i \leq M_m$, and suppose these intervals are disjoint.
2. Let $E_m = \bigcup_{1\leq i\leq M_m} I_{m,i}$; it is also required that $E_m \supseteq E_{m+1}$.
3. Let $K = \bigcap_{m\in\mathbb{N}} E_m$.

We can further assume that K has Lebesgue measure 0, to avoid triviality. Orey and Taylor proved the following theorem for such sets:

Theorem 3.5. *Suppose that $c > 0$, $\delta > 0$. The $\mathrm{meas}_{\beta} K > 0$ if, for every interval $J \subseteq [0,1]$ with $|J| < \delta$, there is a finite integer $m(J)$ such that*

$$M_m(J) \leq c|J|^{\beta} M_m \text{ for } m \geq m(J),$$

where $M_m(J)$ denotes the number of intervals $I_{m,i}, 1 \leq i \leq M_m$ contained in J.

Clearly, each of the intervals used in the construction gives rise to some infinitesimals, depending on the rate at which their lengths converge to 0. In the case of the triadic Cantor set, these are all the same, which makes the construction so simple.

We can therefore consider a Cantor-like set as some union of infinitesimals. The β-Hausdorff measure will not be as simple to calculate as the Cantor set, since we cannot base the hyperfinite time line on a single one of the infinitesimals.

The above theorem in the nonstandard context is actually a restricted version of Frostman's Lemma. For the sake of completeness, we now present Frostman's theorem [21], an important tool in the study of capacity and Hausdorff dimension. The proof itself is so simple that little extra insight is gained from a nonstandard approach. But first we have to define the notion of capacity:

Definition 3.1. *A compact set $A \subset \mathbb{R}^d$ is said to have a positive capacity if it carries a positive measure μ such that the following integral is finite:*

$$I_\alpha(\mu) = \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}.$$

If there is no such measure, we say that A has capacity 0. The capacitarian dimension of A is

$$\dim_C(A) = \sup\{\alpha : \mathrm{Cap}_\alpha A > 0\} = \{\alpha : \mathrm{Cap}_\alpha A = 0\}.$$

In a sense, this is the exact opposite of Hausdorff measure, in that the integral is finite exactly where $\mathrm{meas}_\alpha A$ is infinite, and infinite where $\mathrm{meas}_\alpha A$ is 0. This notion is made precise in the following:

Theorem 3.6. *Let A be a compact set in $\mathbb{R}^d$ and $0 < \alpha < \beta < d$. Then*

$$meas_\beta A > 0 \Rightarrow Cap_\alpha A > 0 \Rightarrow meas_\alpha A > 0.$$

This implies that the capacitarian and Hausdorff dimensions of A are the same.

Proof. First we show that $\mathrm{meas}_\beta A > 0 \Rightarrow \mathrm{Cap}_\alpha A > 0$. If $\mathrm{meas}_\beta A > 0$, A carries a measure μ such that

$$\int_{|x-y|\leq\rho} d\mu(x) < C\rho^\beta$$

for every $y \in \mathbb{R}^d$ and $\rho > 0$, dependant only on μ. Integrating on spherical rings,

$$\int \frac{d\mu(x)}{|x-y|^\alpha}$$

is uniformly bounded with respect to y, and therefore $I_\alpha(\mu) < \infty$, implying $\mathrm{Cap}_\alpha A > 0$.

We now show that $\mathrm{Cap}_\alpha A > 0 \Rightarrow \mathrm{meas}_\alpha A > 0$. If $\mathrm{Cap}_\alpha A > 0$ there is a $\mu \in M^+(A)$ such that $I_\alpha(\mu) < \infty$. Let A_t be the subset of A defined by

$$y \in A_t \text{ if and only if } \int \frac{d\mu(x)}{|x-y|^\alpha} \leq t.$$

For large enough t, $\mu(A_t) > 0$. Let $A \subseteq \cup_n B_n$ such that $A_t \cap B_n \neq \emptyset$ for each n. Choose $y_n \in A_t \cap B_n$. Then

$$\mu(B_n) \leq |B_n|^\alpha \int_{B_n} \frac{d\mu(x)}{|x-y_n|^\alpha} \leq t|B_n|^\alpha \quad \text{and} \sum |B_n|^\alpha \geq t^{-1}\mu(A_t),$$

implying that $\mathrm{meas}_\alpha A_t > 0$ and therefore $\mathrm{meas}_\alpha A > 0$. □

Note that this theorem and Frostman's lemma, are also valid when A is σ-compact instead of compact.

4 Some applications to the fractal geometry of Brownian motion

In this chapter we discuss a nonstandard version of Brownian local time, level sets and the effect of a Brownian motion on a set with a given dimension. Although these results are not original, the proofs using a nonstandard version of Hausdorff dimension are very simple and intuitive. We start with a discussion on Brownian motion in the nonstandard context, with emphasis on Anderson's simple and beautiful construction [2].

4.1 Anderson's construction of Brownian motion

The idea is to construct Brownian motion as a hyperfinite random walk, instead of, as is often done, a limit of random walks. We start with a hyperfinite time line $\mathbf{T}$, based on a fixed $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. We let $\Omega = \{-1, +1\}^{\mathbf{T}}$. If $\omega \in \Omega$, we define the hyperfinite random walk as a polygonal path, filled in linearly between time points $t \in \mathbf{T}$ with $B(\omega, 0) = 0$ and

$$B(\omega, t + \triangle t) - B(\omega, t) = \triangle B(t) = \omega(s)\sqrt{\triangle t},$$

where $\omega(s) = \pm 1$. We let $\mathcal{C}_N$ be the set of all such paths, $\mathcal{A}_N = {}^*\mathcal{P}(\mathcal{C})_N$ and W_N the counting probability on $\mathcal{C}_N$. This gives us the internal probability space

$$(\mathcal{C}_N, \mathcal{A}_N, W_N)$$

which in turn gives us the Loeb space

$$\Omega = (\mathcal{C}_N, L(\mathcal{A}_N), P_N = (W_N)_L).$$

The following theorem is due to Anderson [7]. Recall that an internal function F is S-continuous if, whenever arguments x and y are infinitesimally close, the corresponding function values $F(x)$ and $F(y)$ are infinitesimally close as well.

Theorem 4.1. 1. *For almost all $B \in \mathcal{C}_N$, B is S-continuous and gives a continuous path $b = {}^\circ B \in \mathcal{C}$.*

2. *For Borel $D \subseteq \mathcal{C}$,*

$$W(D) = P_N(st^{-1}(D))$$

is Wiener measure.

3. *The following process is a Brownian motion on the space Ω:*

$$b(t, \omega) = {}^\circ B(w, t) : [0, 1] \times \Omega \to \mathbb{R}.$$

For a proof, as well as a nonstandard version of the central limit theorem, see [1].

4.2 Brownian local time

The local time of a Brownian motion gives a measure of the time a Brownian motion spends at x. The Lebesgue measure of this set is 0, but it can be described using Hausdorff measure, as we shall see shortly.

Definition 4.1. *We define the local time $l(t,x)$ as*

$$l(t,x) = \int_0^t \delta(x - b(s))ds,$$

where b is a Brownian motion and δ the delta function.

The integral therefore "counts" how many times the Brownian path visits x up till the time t. The standard approach (which can be found in detail in [**?**]) is to show there exists a jointly continuous process $l(t,x)$ such that

$$l(t,x) = \frac{d}{dx}\int_0^t I_{(-\infty,x]}(b(s))ds,$$

for almost all $(t,x) \in [0,1] \times \mathbb{R}$, where I_A is the characteristic function of the set A. Note that although the definition is valid for a time line $[0,\infty)$ as well as $[0,1]$, we use a bounded interval throughout. The nonstandard approach, due to Perkins [1], is clearer and more intuitive. We think of the Brownian path b as the standard part of a hyperfinite random walk. The following exposition follows [1]. We start by approximating $l(t,x)$ by

$$(\triangle\, x)^{-1}\int_0^t I_{[x,x+\triangle x]}(b(s))ds.$$

Now replace the time line $[0,1]$ by a discrete hyperfinite time line $\mathbf{T}$ and the space $\mathbb{R}$ by $\Gamma = \{0, \pm\sqrt{\triangle\, t}, \ldots, \pm n\sqrt{\triangle\, t}, \ldots, \pm N\sqrt{\triangle\, t}\}$ and define the internal process $L : \mathbf{T} \times \Gamma \to {}^*\mathbb{R}$ by

$$L(t,x) = \sum_{s<t} I_x(B(s))(\triangle\, t)^{1/2}.$$

Perkins showed that $L(t,x)$ has a standard part which is Brownian local time. He used the nonstandard formulation to prove the following global characterisation of local time, which was previously known to hold only for each x separately: Let $\lambda(t,x,\delta)$ be the Lebesgue measure of the set of points within a distance of $\delta/2$ of $\{s \le t | b(s) = x\}$. Then for almost all $\omega \in \Omega$ and each $t_0 > 0$,

$$\lim_{\delta\to 0^+} \sup_{t\le t_0, x\in\mathbb{R}} |m(t,x,\delta)\delta^{-1/2} - 2(2/\pi)^{1/2} l(t,x)| = 0.$$

It is shown in [**?**] that local time is the same as $\frac{1}{2}$-dimensional Hausdorff measure. From the nonstandard formulation, however, it is immediately clear. If we define the set A as the set of all $t \in [0,1]$ such that $b(t) = x$, the nonstandard local time becomes simply $|A_{\mathbf{T}}| \vartriangle t^{1/2}$. But this is exactly the nonstandard formulation of $\frac{1}{2}$-dimensional Hausdorff measure (up to a finite constant factor — which depends on which author you read) We must now show that level sets have dimension 1/2. We just show this for $x = 0$, since they all have similar dimension. Denote the zero set of a Brownian path $b(\omega)$ by A_ω. We now turn to a standard property of local time to show that the dimension of this set is 1/2. It can be shown (as for instance in [**?**]) that local time is identical in law to the function

$$M_\omega(t) = \max_{s \leq t} b(s).$$

This implies that $P[l(1,0) > 0] = 1$. By the nonstandard formulation of local time, this immediately implies that $\dim A \leq 1/2$, almost surely. By the same token, however, $l(1,0)$ is almost certainly finite, implying that $\dim(A) = 1/2$. The following lemma will be used in the subsequent section. In this case the standard approach is easier than the hyperfinite, by using the Hölder condition for Brownian motion.

Lemma 4.2. *If A is a level set and $D \subseteq A$, then D has dimension $1/2$ or 0.*

Corollary 4.3. *If $\dim A < 1/2$, then the inverse image of any element in $b(A)$ (where b is a Brownian motion) has dimension 0.*

4.3 The image of a set under Brownian motion

A very interesting property of Brownian motion is what it does to sets of a certain Hausdorff dimension. If a compact subset of $[0,1]$ has dimension $\alpha < 1/2$, its image under Brownian motion is a set of dimension 2α. (This set is a Salem set as well, meaning it has equal Hausdorff and Fourier dimensions. This notion will be explored in more depth in Chapter 6.) A set of dimension $\alpha > 1/2$ will have dimension 1 and will almost surely contain an interval. As for sets of dimension 1/2, we have seen above that they may have an image of dimension 0. No hard and fast rule exists for such sets. We now look at nonstandard proofs of these results. The advantage of this approach is a more intuitive (counting) argument. The following was first proved by Kaufman [21].

Theorem 4.4. *Let $A \subset [0,1]$ be a compact set. If $\dim A = \alpha < 1/2$ and b is a Brownian motion, $\dim b(A) = 2\alpha$.*

Proof. The basis for the time line of the image is no longer $\vartriangle t$, but $\sqrt{\vartriangle t}$. Since $|B(A_T)| \leq |A_T|$ and we know that $|A_T| \vartriangle t^\beta \approx 0$ for $\beta > \alpha$, we will have that $|B(A_T)| \vartriangle t^\beta \approx 0$ for any $\beta > \alpha$. Therefore, $|B(A_T)|(\sqrt{\vartriangle t})^\gamma \approx 0$ for $\gamma > 2\alpha$ and we conclude that $\dim b(A) \leq 2\dim A$ (because of the continuity of the functions involved we can conclude that $|[b(A)]_T| = |B(A_T)|$). It is left to show that $\dim b(A) \geq 2\dim A$. This is not quite as simple as the previous proof, since the matter of possible level sets complicates the question of the cardinality of the image. We overcome this by considering only the first elements of level sets and discarding the rest. The remaining set will have the same dimension as the original and the image will have the same cardinality. This is made possible because the set A has a dimension of less than $1/2$. Any subsets of level sets in A are small enough to be left out (mostly) without affecting the dimension. For any $x \in b(A)$, let L_x denote the part of the level set of x contained in A and let $\mathbf{L}$ denote the collection of all these sets. We want to show now that the standard parts of the sums

$$\sum_{L_x \in \mathbf{L}} \frac{1}{N^\alpha}, \quad \sum_{L_x \in \mathbf{L}} \frac{|L_{x,\mathbf{T}}|}{N^\alpha}$$

are 0 and ∞ for the same values of α. To do this, all that is necessary is to show that the first one is infinite whenever the second one is. So suppose that

$$\circ\left(\sum_{L_x \in \mathbf{L}} \frac{|L_{x,\mathbf{T}}|}{N^\alpha}\right) = \infty.$$

We know that

$$\frac{|L_{x,\mathbf{T}}|}{N^\alpha} = s_x^\beta \approx 0$$

for any $\beta > 0$. This implies that

$$\sum_{L_x \in \mathbf{L}} \frac{s_x^\beta N^\beta}{N^\alpha} = \sum_{L_x \in \mathbf{L}} \frac{s_x^\beta}{N^{\alpha-\beta}} \leq \sum_{L_x \in \mathbf{L}} \frac{1}{N^{\alpha-\beta}} = \infty,$$

for β arbitrarily close to 0. Thus we may conclude that the number of level sets is important and not the cardinality of each. But the number of level sets is equal to the cardinality of the range, thus the standard parts of

$$\frac{|B(A_T)|}{N^\alpha} \text{ and } \frac{|A_T|}{N^\alpha}$$

are 0 and ∞ for the same values of α. Keeping in mind that the time line of the image is based on $\sqrt{\triangle t}$ and not $\triangle t$, we can conclude that the dimensions are equal.

5 Nonstandard analysis of rapid points of functions

Orey and Taylor found the exact Hausdorff dimension of rapid points of Brownian paths [31]. Because of the importance of the result and the beauty of its proof (utilising Cantor-type sets), we present a sketch of the proof in the appendix. We show in this chapter that the result holds for a more general class of functions satisfying a few conditions which hold almost surely for Brownian paths. Specifically, any function that satisfies these conditions will have a set of rapid points of a given dimension, and Brownian motion satisfies these conditions with probability 1.

First, we need a some definitions. Given an interval $I = [a, b]$, we define

$$R_f(I) = \sup_{a \leq s < t \leq b} |f(t) - f(s)|.$$

When it is clear which function we are using, we usually write just $R(I)$. For a function f and a given $0 < \alpha < 1$, let $\mathcal{I} = \mathcal{I}(\alpha)$ denote the collection of all intervals I in $[0, 1]$ such that $R(I) > \alpha\sqrt{2h \log h^{-1}}$, where $h > 0$ denotes the length of the interval I. Given such an I, we denote by I_k the collection of all the intervals of the form $[i2^{-k}, (i+1)2^{-k})$ (for $i = 0, 1, \ldots, 2^k - 1$) contained in $\mathcal{I}$. The set of α-rapid points of Brownian motion, E_α, is defined in (1.4).

We first turn to a requirement which will allow certain sets to have a dimension of, or more than, $1 - \alpha^2$.

Lemma 5.1. *Suppose* $0 < \alpha < 1$. *Let* f *be a continuous function and consider an equipartition of* $[0, 1]$ *into* 2^n *pieces, each of which is further subdivided in a further* 2^j *pieces. If there exists some* $c > 0$*, dependant only on* f*, such that the relation*

$$|\{0 \leq k \leq 2^n - 1 : \exists t \in [k2^{-n-j}, (k+1)2^{-n} + 2^{-j}](2^{n/2}|f(k+1)2^{-n}) - f(t)| \geq \alpha\sqrt{2n \log 2}\}| \geq c2^{(1-\alpha^2)(n+j)} \quad (5.1)$$

is satisfied for large enough n*, the* α*-rapid points of* f *have dimension larger than or equal to* $1 - \alpha^2$.

Although the formulation may seem somewhat cumbersome, we do it as such to facilitate the application to Brownian motion at a later stage.

Proof. Consider the relation

$$|\{0 \leq k \leq 2^{n+j} - 1 : \exists t \in [kb2^{-n}, (k+1)b2^{-n}](2^{n/2}|f(k2^{-n} + 2^{-n}) - f(t))| \geq \alpha\sqrt{2n \log 2}\}| \geq c2^{(1-\alpha^2)(n+j)}. \quad (5.2)$$

Everything in this relation is first order and can be transferred to a hyperfinite context; it follows that there are infinite numbers N and J for which the relation also holds; for convenience we set $M = N + J$:

$$|\{1 \leq K \leq 2^M : 2^{N/2}|F((K+1)2^{-N}) - F(T))\}| \geq \alpha\sqrt{2M \log 2}\}| \geq c2^{(1-\alpha^2)M}.$$

(F is the S-continuous nonstandard extension of f; see Section 2.3.) Now, instead of seeing the division of $[0,1]$ as an equipartition, we can consider it a hyperfinite time line. Also, remembering the ultrapower construction, each K for which the above holds implies the existence of a sequence of dyadic rationals which converges to a rapid point. The hyperfinite dyadic rationals included in the transferred relation therefore exist in the monad of an α-rapid point. But we know that there are $\geq c2^{(1-\alpha^2)N}$ points on our time line of 2^N elements. To let the quotient

$$\frac{|E_\alpha|}{N^\beta}$$

therefore have real part 0, N would have to be raised to a power larger than $1-\alpha^2$, for the quotient would then be a constant. Thus, $\dim E_\alpha \geq 1-\alpha^2$. □

We now confirm that Brownian motion does indeed have these properties, almost surely, and that the α-rapid points of Brownian motion therefore have dimension $1-\alpha^2$. In the following two lemmas we assume we are working with a Brownian motion X on a space $(\Omega, \mathcal{B}, \mathbf{P})$.

Lemma 5.2. *If A is the set of α-rapid points of $X(t)$, A has a Hausdorff dimension of at most $1-\alpha^2$.*

Proof. We consider a partial covering of E_α by dyadic intervals not unlike those considered in the previous lemma. Since each member of E_α can be approached through dyadic rationals, this will indeed form a cover in the limit. Let $n, j \in \mathbb{N}$ and let $\alpha_1 < \alpha$. We will consider j to be fixed. Define $B_{\alpha_1,n}(\omega)$ to be the random set

$$\{0 \leq k \leq 2^{n+j}-1 : \exists t \in [k2^{-n-j},(k+1)2^{-n-j}](2^{n/2}|f(k2^{-n}+2^{-n})-f(t)| \\ \geq \alpha_1\sqrt{2n\log 2}\}$$

Note that we can either consider these sets as subsets of the integers or as collections of the dyadic intervals these integers represent. We shall use these interchangeably, since it will always be clear from the context which we mean. Let $A_{\alpha_1,n}$ be the event

$$\{|B_{\alpha_1,n}(\omega)| \geq 2^{(n+j)(1-\alpha_1^2)}\}.$$

The sets of the form $B_{\alpha_1,n}(\omega)$ do not form a cover of the rapid points, but it is easily seen that the α-rapid points are contained in the limit superior of such sets, indexed by n.

We now estimate the probability of $A_{\alpha_1,n}$. The distribution of $A_{\alpha_1,n}$ is binomial and the probability of a success is calculated in Theorem 6.4 to be larger than $2^{-\alpha_1^2 n}2^{o(1)}$. We now want to calculate the probability $\mathbf{P}(A_{\alpha_1,n})$. For this we use an estimate from [10] for the tail of the binomial distribution. If S_{2^n} denotes the sum of 2^n variables which may take value 1 with probability p and 0 with probability $q=1-p$, then we have that

$$\mathbf{P}\{S_{2^n} \geq r\} \leq \frac{rq}{(r-2^n p)^2}, \tag{5.3}$$

when $r > 2^n p$. To see that we may use this estimate, using the approximations in the proof of Lemma 6.4, we find that $p < 2^{-n\alpha_1^2(1+o(1))}$ (using Feller's estimate of the maximum fluctuation over an interval as upper bound). We let the constant c be determined by j in the form $c \geq 2^{-j\alpha^2}$. Then it will be true that $r > np$, for fixed j. Using the lower bound for p it is clear that

$$\begin{aligned} p &\geq 2^{-\alpha_1^2 n} 2^{\mathrm{o}(1)} \\ &> 2^{-\alpha_1^2(n+j)} > \frac{2^{-m\alpha_1^2}}{m}, \end{aligned} \tag{5.4}$$

where $m = n + j$. Our estimate becomes

$$\mathbf{P}(A_{\alpha_1,n}) < \frac{2^{m(1-\alpha_1^2)}(1-p)}{(2^{m(1-\alpha_1^2)} - 2^m p)^2}.$$

through the use of (5.4), and using (5.5) we obtain

$$\mathbf{P}(A_{\alpha_1,n}) \leq \frac{2^{m(1-\alpha_1^2)}(1 - 2^{m\alpha_1^2}/m)}{(2^{m(1-\alpha_1^2)} - 2^m p)^2}.$$

Not only can some quick calculation show that this term tends to zero as n tends to infinity, but we also have that the sum of all the terms converges, because of the inequalities

$$\begin{aligned} \mathbf{P}(A_{\alpha_1,n}) &< \frac{2^{m(1-\alpha_1^2)}(1 - 2^{m\alpha_1^2}/m)}{(2^{m(1-\alpha_1^2)} - 2^m p)^2} \\ &< \frac{2^m(m2^{-m\alpha_1^2} - 2^{m(\alpha_1^2 - \alpha 1^2)})}{m(2^{m(1-\alpha_1^2)} - 2^m)^2} \\ &\leq \frac{2^{(1-\alpha_1^2)m}}{m(2^{m(1-\alpha_1^2)} - 2^m)^2} \\ &< 2\frac{2^{(1-\alpha_1^2)m}}{2^m} \quad \text{since } (2^{-\alpha_1^2 m} - 1)^2 > \frac{1}{2} \text{ for } m \text{ large} \\ &= \frac{2}{2^{m(1+\alpha_1^2)}}. \end{aligned}$$

Seeing the above as the first step in constructing our cover, we can now proceed to larger values of n and α_1 to find intervals of smaller diameter. For such larger values the above inequalities will still hold. We also consider, for each larger value of n, a larger value α_i, where $\alpha_1 < \alpha_{i-1} < \alpha_i < \alpha$. If we now consider, for a specific sequence $\{\alpha_i\}_{i\in\mathbb{N}}$, the collection of intervals given by all the $B_{\alpha_1,n}$, we obtain a cover of A. Although we have constructed the sets as unions of closed intervals, they may as well be considered to be made up of open intervals, since the set of dyadic rationals which are also rapid points have Hausdorff dimension 0. Although the compactness of the set ensures that we could find a finite subcover, we do not actually need to find such a cover here, since the number of intervals used is small enough. If we now consider the α_1-Hausdorff sum for the cover of A obtained by the above process, we get an expression smaller than

$$\begin{aligned} & \sum |B_{\alpha_i,n}| 2^{1-\alpha_1^2(n+j)} \\ < & \sum 2^{(n+j)(1-\alpha_i^2)} 2^{(-1-\alpha_1^2)(n+j)} \\ = & 2^{(\alpha_1^2-\alpha_i^2)(n+j)} \end{aligned}$$

which is bounded, with an exceptional probability which is as close to 0 as we wish (since the sum of the probabilities of each exceptional case can be made arbitrarily small by simply starting our construction at a larger value of n). This holds for all the $1-\alpha_i^2$-Hausdorff sums. Since A is contained in the set whose dimension we are now approximating, it must follow that dim$A \leq 1-\alpha^2$, with probability 1. □

Lemma 5.3. *There exists a constant $c < 1$ such that Brownian motion almost surely satisfies relation* 5.2*; that is, for large enough n,*

$$|\{0 \leq k \leq b2^n - 1 : \exists t \in [kb2^{-n}, (k+1)2^{-n} + 2^{-j}](2^{n/2}|f(k+1)2^{-n}) - f(t)| \\ \geq \alpha\sqrt{2n\log 2}\}| \geq c2^{(1-\alpha^2)(n+j)}. \quad (5.5)$$

Proof. We again use a binomial distribution on the set of intervals, viewing it as a Bernoulli trial with probability of success p (as calculated in Theorem 6.5) and using another estimate of the binomial tail from [10], we find that

$$\begin{aligned} \mathbf{P}\left(\left\{S_m \leq \frac{1}{2}2^{(1-\beta^2)m}\right\}\right) & \leq \frac{(m-r)p}{(mp-r)^2} \\ & = \frac{(2^m - \frac{1}{2}2^{(1-\beta^2)m})2^{-\beta^2 m}2^{o(1)}}{(2^{(1-\beta 2)m}2^{o(1)} - \frac{1}{2}2^{(1-\beta^2)m})^2} \\ & = \frac{2^{o(1)}(2^{(1-\beta^2)m} - \frac{1}{2}2^{(1-\beta^2)m})}{(2^{o(1)}2^{(1-\beta 2)m} - \frac{1}{2}2^{(1-\beta^2)n})^2}. \end{aligned}$$

This clearly tends to 0 and thus the probability of more than $\frac{1}{2}2^{(1-\beta^2)m}$ successes in 2^m trials goes to 1, which proves the lemma. □

It now follows trivially from the previous two lemmas that the α-rapid points of a Brownian motion have almost surely a Hausdorff dimension of $1-\alpha^2$.

This theorem has the following interesting (but simple) result as consequence [31]:

Corollary 5.4. *For a Brownian path $X_\omega = X$,*

$$dim\left\{t : \limsup_{h\to o} \frac{X(t+h) - X(t)}{(2h\log\log h^{-1})^{\frac{1}{2}}} = \infty\right\} = 1$$

with probability 1.

Proof. It is easily seen that that for each α, the set of α-rapid points has the property of the above set, with probability 1 (the iterated logarithm is too weak to "contain" the growth at the rapid points). The above set therefore contains all the $E(\alpha)$ and has dimension 1, with probability 1. $\square$

We also remark that the theorem can quite easily be extended to higher dimensions, with the most notable change being the replacement of $X(t+h) - X(t)$ by the Euclidean distance $|X(t+h) - X(t)|$.

As mentioned in the introduction, the notable feature of this version of the proof is the pathwise approach it takes. In the next section we will repeatedly use the probability that a section contains an α-rapid point, approximated by $h^{\alpha^2} h^{o(1)}$, where h is the length of the interval. This is very close to our approximation of the ratio of intervals which are picked at any stage. The probability of the paths having a certain property is therefore somehow reflected in *each path*. This ratio is used in Section 6.2 to find upper and lower bounds on the total mass of a measure, while the probability will lead us to the domain of validity of the measure. As we shall see, these two in combination give us a *Salem set*.

6 The Fourier dimension of rapid points

6.1 Introduction

In this chapter we explore a subtler property of the rapid points of Brownian motion than their Hausdorff dimension. The main result of this chapter is attributable to Kaufman [22], but since his proof is perhaps not entirely transparent (and sometimes inaccurate) we feel it is worthwhile giving an exposition thereof.

The Hausdorff dimension of a set gives us a measure of the "thinness" of the set. As we shall show shortly, when it is regarded as a capacitarian dimension, it can also yield information on the asymptotic behaviour of the Fourier transform of measures supported by the set.

Given a measure μ on our usual set $[0,1]$, we define the Fourier transform of μ as

$$\hat{\mu}(\xi) = \int_{[0,1]} e^{i\xi x} d\mu(x).$$

We then have the following result, which is a statement about capacity in terms of Fourier transforms:

Theorem 6.1. *If $A \subseteq \mathbb{R}^d$ is a compact set, then the dimension of A is the supremum of all α for which there exist a positive measure μ such that*

$$\int_{\mathbb{R}^n} |\hat{\mu}(\xi)|^2 |\xi|^{\alpha-d} d\xi < \infty.$$

Proof. Recall from Definition 3.1 that a compact set $A \subset \mathbb{R}^d$ is said to have a positive capacity if it carries a positive measure μ such that the following integral is finite:

$$I_\alpha(\mu) = \iint \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}.$$

It follows from the Fourier analysis of Schwartz distributions that there is a constant $C = C(d,\alpha)$ such that

$$I_\alpha(\mu) = C \int_{\mathbb{R}^d} |\hat{\mu}(\xi)|^2 |\xi|^{\alpha-d} d\xi,$$

as for example on p.162 of [29]. The result follows. $\square$

Sometimes more precise information about $\hat{\mu}$ is required. In harmonic analysis, one of the most important problems is that of the uniqueness and multiplicity of sets. A compact set $E \subseteq \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is called a *set of (restricted) multiplicity* if there is a measure supported on $\mathbb{T}$ such that the trigonometric series

$$\sum_{n\in\mathbb{Z}} \hat{\mu}(n) e^{2\pi i n x}$$

vanishes everywhere outside E. These sets are sometimes called M_0-sets. The concept can be extended:

Definition 6.1. *An M_β-set is a compact set in $\mathbb{R}^n$ which carries a measure μ such that $\hat{\mu}(\xi) = o(|\xi|^{-\beta})$ as $|\xi| \to \infty$. For a compact set E, we call the supremum of the α such that E is an $M_{\alpha/2}$-set, the* Fourier dimension *of E. We shall denote this by $dim_F E$.*

The Fourier dimension is a somewhat more elusive character than Hausdorff dimension ($\dim E$). They are usually different, as for instance in the case of the triadic Cantor set, which has positive Hausdorff dimension but a Fourier dimension of 0 [24]. The following lemma shows that the dimensions are related; specifically, that the Fourier dimension is majorised by the Hausdorff dimension.

Lemma 6.2. *For a compact set $E \subseteq \mathbb{R}^d$, $dim E_F \leq dim E$.*

Proof. We use the previous theorem. Suppose that $\dim_F E > 0$. Choose $0 < \alpha < \beta < \dim_F A$. Then there exists some $\mu \in M^+(E)$ (the set of strictly positive measures on E) with $|\hat{\mu}(\xi)|^2 = \mathrm{o}(|\xi|^{-\beta})$ as $\xi \to \infty$. In addition there are constants C and C_1 such that

$$\begin{aligned} I_\alpha(\mu) &= C \int |\hat{\mu}(\xi)|^2 |\xi|^{2-d} d\xi \\ &\leq C_1 \int_{|\xi| \geq 1} \frac{d\xi}{|\xi|^{d+(\beta-\alpha)}} + \mathrm{O}(1). \end{aligned}$$

Thus, $\mathrm{Cap}_\alpha E > 0$, and $\dim E \geq \dim_F E$. □

One of the interesting aspects of Fourier, as opposed to Hausdorff, dimension is its arithmetical properties, as illustrated by the following theorem.

Theorem 6.3. *If E has a strictly positive Fourier dimension, then the abelian group generated by E is the entire ambient Euclidean space $\mathbb{R}^n$.*

Proof. We use the following result of Steinhaus [35] in the proof: If $E \subseteq \mathbb{R}^n$ is such that its Lebesgue measure exists and is non-zero, then for some $\varepsilon > 0$,

$$B(\varepsilon) = \{x : \|x\| < \varepsilon\} \subset E - E,$$

where $E - E$ is defined as the set $\{x - y : x, y \in E\}$.

Let $\alpha > 0$ be such that for some $\mu \in M^+(E)$,

$$|\hat{\mu}(\xi)|^2 = \mathrm{o}(|\xi|^{-\alpha}) \text{ for } |\xi| \geq 1.$$

Set $\nu = \mu \star \cdots \star \mu$, the convolution product of μ with itself, k times, where k is large enough that $k\alpha > n$. From the well-known properties of convolution we have

$$|\hat{\nu}(\xi)|^2 = |\hat{\mu}(\xi)|^{2k} = \mathrm{o}(|\xi|^{-k\alpha}),$$

implying that $\mathrm{supp}\nu$ has positive Lebesgue measure (through the use of Parseval's theorem). But $\mathrm{supp}\nu \subset \mathrm{supp}\mu + \cdots + \mathrm{supp}\mu$ (k times) and therefore, for some $\varepsilon > 0$,

$$B(\varepsilon) \subset (E + \cdots + E) - (E + \cdots + E)$$

(the difference of E summed k times and itself). Since $\mathbb{R} = \bigcup_{n\geq 1} nB(\varepsilon)$, we can conclude that the abelian group generated by E is $\mathbb{R}^n$. $\square$.

When the Fourier and Hausdorff dimensions of a set coincide, it is called a Salem set, after Raphael Salem. The first (nontrivial) set known to have this property almost surely was constructed by Salem [34]. Trivial examples of such sets are the balls in any dimension; for example, the interval $[0,1]$ has both dimension equal to one (as a subset of $\mathbb{R}$), and the appropriate measure in both cases is the Lebesgue measure. For quite a while the only Salem sets were random sets. Gatesoupe [14] showed how to construct Salem sets of arbitrary dimension $0 < \alpha < d$ for $d > 1$ by using rotations, given Salem sets in $\mathbb{R}$ of Fourier dimension β, $0 < \beta < 1$. Only in 1981 did Kaufman [23] devise a way to construct Salem sets of arbitrary dimension $0 < \alpha < 1$, with certainty. In 1996 C. Bluhm [5] expanded this construction. We give a summary of Bluhm's result in the following section, since in section 6.2 we will consider the probabilistic construction of Salem sets. Also worth mentioning is the result of Rudin [33], using a set introduced by Salem: There exist M_0-sets which are independent in $\mathbb{R}^d$ over the rationals. Such a set carries a pseudomeasure which is not a measure, but all discrete measures it carries have the same norm whether considered as pseudomeasures or measures. For a wide class of thin sets, the Brownian image of the set is a *Rudin set*, i.e. an M_0-set independent over the rationals.

6.1.1 Construction of a deterministic Salem set

First, we fix some notation to be used in this section. For $x \in \mathbb{R}$,

$$\|x\| = \min_{m\in\mathbb{Z}}|x - m|.$$

The set of prime numbers will be denoted by $\mathbb{P}$, and we let $\mathbb{P}_M = \mathbb{P} \cap [M, 2M]$. The inspiration for the construction is the set of α-well-approximable number, defined as

$$F(\alpha) = \bigcap_{k=1}^{\infty} \bigcup_{q=k}^{\infty} \{x \in [0,1] : \|qx\| < q^{-1-\alpha}\}.$$

The Hausdorff dimension of this set is $2/(2+\alpha)$. This has been known for quite some time, since the work of Jarnik [20] and Besicovitch [3]. Although the set S_α to be defined momentarily resembles $F(\alpha)$ strongly, the set of α-well-approximable number is dense in $[0,1]$, which S_α is not. Also used in the paper of Bluhm is the prime number theorem in the following by Hardy and Wright [15]:

$$\lim_{M\to\infty} \frac{\#\mathbb{P}_M}{M/\log M} = 1$$

where $\#\mathbb{P}_M$ denotes the number of prime numbers smaller than M. For the Cantor-type construction that is used, a sequence of integers $(M_k)_{k\in\mathbb{N}}$ is constructed recursively, satisfying

$$M_1 < 2M_1 < M_2 < 2M_2 < M_3 < 2M_3 < \cdots .$$

Furthermore, according to the prime number theorem we can find such a sequence which also satisfies, for every $k \in \mathbb{N}$,

$$\mathbb{P}_{M_k} \neq \emptyset \quad \text{and} \quad \#\mathbb{P}_{M_k} \geq M_k/(2\log M_k).$$

We now consider the following set:

$$\bar{F}_q(\alpha) = \{x \in [0,1] : \|qx\| \leq q^{-1-\alpha}\}.$$

We conclude that the set is compact by writing it as a union of closed intervals:

$$\bar{F}_q(\alpha) = [0, q^{-2-\alpha}] \cup \left(\bigcup_{m=1}^{q-1} \left[\frac{m}{q} - q^{-2-\alpha}, \frac{m}{q} + q^{-2-\alpha} \right] \right) \cup [1 - q^{-2-\alpha}].$$

The set which then carries an appropriate measure for it to be a Salem set of dimension $2/(2+\alpha)$ is defined by

$$S_\alpha = \bigcap_{k=1}^{\infty} \bigcup_{p \in \mathbb{P}_{M_k}} \bar{F}_p(\alpha).$$

The nonzero measure μ_α constructed by Bluhm on the set S_α satisfies the decay condition

$$\hat{\mu}_\alpha(x) = o(\log|x|)|x|^{\frac{-1}{2+\alpha}}, \quad |x| \to \infty.$$

6.1.2 The occurrence of Salem sets in Brownian motion

Thus far it may appear that Salem sets are extremely rare and possibly of little intrinsic value. However, once we turn to random sets, specifically those associated with Brownian motion, Salem sets seem to arise naturally and often. For instance, consider any subset of $[0,1]$ of Hausdorff dimension (strictly) between 0 and $1/2$. We have seen in Chapter 4 that the dimension doubles under the action of Brownian motion; perhaps even more interestingly, the set has become a Salem set [21]. When the dimension of the original set is larger than $1/2$, its dimension clearly cannot be doubled, since both Hausdorff and Fourier dimensions are bounded by 1. The image does however attain this maximum, and what is more, will contain a set of positive Lebesgue measure [21] (clearly, Brownian motion does not respect topological dimension either). For a thorough study of Brownian images and Salem sets, there is a whole series of (difficult) papers by Kaufman, as listed on p.289 of [21].

There is another instance where the Brownian motion brings forth Salem sets which we will examine closely in the next section (again, due to Kaufman). We have already concerned ourselves with the Hausdorff dimension of the rapid points. What is more surprising than these properties is the fact that this “encoding” of the dynamics of Brownian motion as a linear set also generates a set with similar Fourier dimension, almost surely.

There are still many questions remaining in the study of Fourier dimensional properties of Brownian motion. It is (to the author's knowledge) not yet known what the Fourier dimension of the zeroset of Brownian motion is, although we have seen in an earlier chapter that it has a Hausdorff dimension of 1/2. We have clearer results for sets of Hausdorff dimension unequal to 1/2, but still do not entirely understand why this dividing line has such an effect on dimension.

6.2 Large increments of Brownian motion

The main result of this section is the following:

Theorem 6.4. *(Kaufman [22]) With probability 1, a certain compact subset of E_α, the α-rapid points of a given Brownian motion X, carries a probability measure μ such that $\hat{\mu}(\xi) = o(\xi^{\frac{1}{2}(\alpha^2-1)}), 1 \leq \xi < \infty$. That is, E_α is a set of Fourier dimension $1-\alpha^2$.*

In proving this we will need a lemma. Suppose that $\{\xi_n; n \geq 1\}$ is a set of independent random variables with a common distribution

$$\mathbf{P}\{\xi_n = 1\} = p = 1 - \mathbf{P}\{\xi_n = 0\}.$$

We want estimates for sums of the form $\sum(p-\xi_n)a_n$. Let $\sigma^2 = \sum |a_n|^2$. We let $B = \max|a_n|$.

Lemma 6.5. *Provided that $YB < 2p\sigma^2$ and all else as above,*

$$\mathbf{P}\{|\sum(p-\xi_n)a_n| \geq Y\} \leq 4\left(\exp -\frac{1}{4}p^{-1}\sigma^{-2}Y^2\right). \tag{6.1}$$

Proof. We use a basic inequality that can be easily obtained by writing out the series expansion of e:

$$pe^{t(1-p)} + (1-p)e^{-pt} \leq 1 + p(1-p)t^2 \leq e^{t^2 p},$$

which is valid for $0 \leq p \leq 1$, $-1 \leq t \leq 1$. We now turn to Chebyshev's inequality, which states that

$$\mathbf{P}\{X \geq k\} \leq \frac{1}{k}\mathbb{E}(X)$$

for any random variable X with finite expectation. Kaufman mentions that he deduces the final inequality by using Chebyshev, but it is never indicated how this should be done. We now give the details required to obtain an estimate of the probability

$$\mathbf{P}\{|\sum(p-\xi_n)a_n| \leq Y\}.$$

Note that this will be the same as the probability

$$\mathbf{P}\{e^{|\sum(p-\xi_n)a_n|} \leq e^Y\}.$$

Set $X = \sum(p - \xi_n)a_n$. Because of the assumed independence of the ξ_n, the expected value of e^{tX} can be evaluated by the integral

$$\begin{aligned}\int e^{tX} d\mathbf{P} &= \prod_{i=1}^{n} \int e^{-(p-\xi_n)a_n t} d\mathbf{P} \\ &= \prod (pe^{-(1-p)a_n t} + (1-p)e^{pa_n t}) \\ &\leq \prod e^{pt|a_n|^2} \\ &= e^{pt\sigma^2}.\end{aligned}$$

Note that we have used the basic inequality repeatedly with $a_n t$ instead of just t, and must require that $0 \leq \max|a_n|t \leq 1$. It then follows that

$$\mathbf{P}\{X \geq Y\} \leq e^{t^2 p\sigma - tY}.$$

By symmetry we can conclude that

$$\mathbf{P}\{|X| \geq Y\} \leq 2\exp(pt\sigma^2)\exp(-tY). \tag{6.2}$$

We now choose a specific value of t to minimise the right-hand side of the inequality. It is easily seen that $t = Y/2p\sigma^2$ is such a value. This yields

$$\mathbf{P}\{|X| \leq Y\} \leq 2\exp\left(-\frac{1}{4}p^{-1}\sigma^{-2}Y^2\right),$$

as long as $YB \leq 2p\sigma^2$. We do still require an inequality for complex values of a_n and can obtain a rough but useful one by considering the above for separate real values and considering the probabilities of $|X| \leq Y/2$, and then doubling the right hand side. We then obtain

$$\mathbf{P}\{|X| \leq Y\} \leq 4\exp\left(-\frac{1}{4}p^{-1}\sigma^{-2}Y^2\right). \square$$

To proceed with the proof of the theorem we need some further inequalities. Let

$$S = \max|X(b) - X(a)|, \quad 0 \leq a < b \leq 1.$$

The exact distribution of S was found by Feller [10], but we need only an approximation of the following form:

$$\mathbf{P}\{S \geq Y\} = \exp(-\frac{1}{2}Y^2)\exp o(Y^2), \quad Y \to \infty.$$

In the original paper it is never mentioned how one arrives at this probability (except for indicating that it is a consequence of the reflection principle and the Gaussian distribution). We obtain this estimate from the distribution of Brownian motion and from the reflection principle as follows: The maximum difference will clearly exceed Y if $M(1)$ (the maximum of a path over $[0,1]$) exceeds Y, since $m(1)$ (the minimum over $[0,1]$) is almost surely less than 0. By André's reflection principle (see, for instance [19]), we have

$$\mathbf{P}\{M(1) \geq Y\} = 2\mathbf{P}\{B(1) \geq Y\}.$$

The probability we want to estimate is clearly larger than this (obviously more than twice as large, again by the reflection principle).
To get an upper bound, consider the two Markov times $\tau_m(\omega)$ and $\tau_M(\omega)$ which signify the $t \in [0,1]$ on which the Brownian path reaches its minimum and maximum for the first time, respectively. (These *passage times* are well-defined; see for instance p.25 of [18].) We create a new time by taking the least of the two:

$$\tau(\omega) = \min(\tau_m(\omega), \tau_M(\omega)).$$

With this Markov time we form a new Brownian motion, which is the old one reflected about τ:

$$B_1(t) = 2B(\tau(\omega)) - B(t).$$

Now, all paths for which $S \geq Y$ will have the property that the corresponding paths of B_1 have maxima or minima exceeding Y. However, all of the reflected paths which have this property do not necessarily have that $S \geq Y$ for the original Brownian motion. The probability of this set therefore provides the upper bound we seek. Explicitly, the probability is given by

$$\begin{aligned}
& \mathbf{P}\{M(1) \geq Y \text{ or } m(1) \leq Y\} \\
= \; & \mathbf{P}\{M(1) \geq Y\} + \mathbf{P}\{m(1) \leq -Y\} - \mathbf{P}\{M(1) \geq Y \text{ and } m(1) \leq -Y\} \\
= \; & 4\mathbf{P}\{X(1) \geq Y\} - \mathbf{P}\{M(1) \geq Y \text{ and } m(1) \leq -Y\} \\
\leq \; & 4\mathbf{P}\{X(1) \geq Y\}
\end{aligned}$$

We have therefore that the desired probability is bounded from above and below by terms which are both $e^{-\frac{1}{2}Y^2}e^{o(Y^2)}$ (since that is also the distribution of the tail of Brownian motion) and can therefore be approximated by the same form.

Suppose now that

$$g(h) = \sqrt{2h \log h^{-1}}$$

Next we want to use this to find the probability of the event

$$X(h) - X(t) \geq (\beta - 2b^{\frac{1}{2}})g(h),$$

where h tends to 0, $0 < \beta < 1$ and b is the reciprocal of an integer (which we will later assume to be large enough for our purposes).
Kaufman obtains this by combining the probabilities of two events. His reasoning, however, is not clear, since the events are very much *not* independent, and

do not necessarily lead to the desired approximation. We now indicate how this probability can be estimated.

We will need the estimate of the tail of Brownian motion in the following form [31]:

$$\mathbf{P}\{X(t+h) - X(t) > \lambda h^{\frac{1}{2}}\} = e^{-\frac{1}{2}\lambda^2(1+o(1))}. \tag{6.3}$$

The desired probability will be larger than the probability of the maximum fluctuation over $[0,h]$ being greater than $\beta g(h)$ and the maximum over $0 \leq t \leq bh$ being less than $2b^{\frac{1}{2}}$. This is given by

$$\mathbf{P}\{S([0,h]) \geq \beta g(h)\} - \mathbf{P}\{M([0,bh]) \geq 2b^{\frac{1}{2}} g(h)\}.$$

Using 6.3, we approximate the first part by

$$\begin{aligned} \mathbf{P}\{S([0,h]) \geq \beta g(h)\} &= \exp\left[\beta^2 \log h + o(\beta^2 \log h^{-1})\right] \\ &= h^{\beta^2}(1+0(1)). \end{aligned} \tag{6.4}$$

In approximating the second part, we use the fact that the probability of $\{X(bh^{\frac{1}{2}}) \geq b^{\frac{1}{2}} Y\}$ is equal to that of $\{X(h) \geq Y\}$ (see Proposition 1.3.2). Setting $Y = 2b^{\frac{1}{2}} g(h)$, we get

$$\begin{aligned} \mathbf{P}\{M([0,bh]) \geq 2b^{\frac{1}{2}} g(h)\} &= 2\mathbf{P}\{X(bh) \geq 2b^{\frac{1}{2}} g(h)\} \\ &= 2\mathbf{P}\{X(h) \geq 2bg(h)\} \\ &= e^{-\frac{1}{2}Y^2} e^{o(Y^2)} \\ &= \exp\left[(4\log h)(1+o(1))\right] \\ &= h^{4(1+o(1))}. \end{aligned} \tag{6.5}$$

Subtracting 6.5 from 6.4, we find a probability of $h^{\beta^2}(1+o(1) - h^4 h^{o(1)})$. In his paper, Kaufman uses the fact that the probability is larger than $h^{\beta^2} h^{o(1)}$. Note that, as the second term in our probability also approaches 1, we may use the same approximation.

We can now start our construction. Let $0 \leq r < s \leq 1$ and let I_n be a division of the interval (r,s) into N equal intervals of length $(s-r)N^{-1}$. We further subdivide each interval I_n into intervals I_n^q of length $(s-r)bN^{-1}$, where we assume that $b^{-1} \in \mathbb{Z}$ and $1 \leq q \leq b^{-1}$. We select I_n^q with lower extremity x if

$$X(x+h) - X(t) \leq (\beta - 2b^{\frac{1}{2}})g(h) \text{ on } x \leq t \leq x + bh,$$

where h is the length of the interval. The selection of the intervals I_n^q ($1 \leq n \leq N$) are mutually independant for each q, with the probability $p = p_N \geq N^{-\beta^2} N^{o(1)}$ for large N, as shown above. Let μ_0 be Lebesgue measure on $[r,s]$ and let ξ be the characteristic function of the selected intervals. For a Borel subset A of $[r,s]$, define

$$\mu_1(A) = p^{-1}\xi(A)\mu_0(A).$$

(We suppose that $\xi(A) = 1$ if there is some $x \in A$ such that $\xi(x) = 1$, and $\xi(A) = 0$ otherwise.)

Lemma 6.6. *For any $\varepsilon > 0$ and large enough N, the inequality*

$$|\hat{\mu_1}(u) - \hat{\mu_0}(u)| < \varepsilon(1+u)^{\frac{1}{2}(\alpha^2-1)}$$

holds for all $u > 0$, with probability approaching 1 as $N \to \infty$.

Proof. Each $q = 1, \ldots, b$ determines a decomposition $\mu_1 = \sum_{q=1}^{b^{-1}} \mu_1^q$ and $\mu_0 = \sum_{q=1}^{b^{-1}} \mu_0^q$. Because b is fixed it suffices to prove the inequality for each pair $\hat{\mu}_1^q$ and $\hat{\mu}_0^q$ (each of which is already a summation over $1, \ldots, N$) and we drop the superscript from now on. By letting $f_n(u)$ be the Fourier transform of Lebesgue measure, we get that

$$\hat{\mu}_0(u) - \hat{\mu}_1(u) = \sum_{n=1}^{N} (1 - p^{-1}\xi_n(u)) f_n(u),$$

where $|f_n(u)| \leq 2|u|^{-1}$. We set $C(u) = \max |f_n(u)|$ and rewrite the sum as in Lemma 6.5:

$$\left| \sum_{n=1}^{N} (p - \xi_n) f_n(u) \right| < \varepsilon p(1+u)^{\frac{1}{2}(\alpha^2-1)}, \tag{6.6}$$

where $B = C(u)$ and $\sigma = NC^2(u)$. We now divide up the positive reals and prove inequality 6.6 for each section.

Suppose $0 < u \leq N$. On this interval we replace $f_n(u)$ with an upper bound for $\max |f_n(u)|$, which is $2N^{-1}$ and σ^2 with the upper bound N^{-1}. Since the new sum is larger, proving the new inequality will imply the original one. We can also easily enough get rid of the factor 2 and replace the upper bound for $\max |f_n(u)|$ by N^{-1}. Now let $Y = \varepsilon p N^{\frac{1}{2}(\alpha^2-1)}$. We are therefore considering the inequality

$$\mathbf{P}\left\{ \left| \sum (p - \xi_n) N^{-1} \right| \geq \varepsilon p N^{\frac{1}{2}(\alpha^2-1)} \right\}. \tag{6.7}$$

Lemma 6.5 is applicable because

$$YB = \varepsilon p N^{-1} N^{\frac{1}{2}(\alpha^2-1)} < pN^{-1} = p\sigma^2.$$

Lemma 6.5 then gives us a probability of less than $4e^{-\frac{1}{2}p^{-1}N^2Y^2}$. Some quick calculation shows that

$$p^{-1}N^2Y^2 \geq c\varepsilon^2 N^{\alpha^2\beta^2-1} N^{o(1)}, \tag{6.8}$$

which is larger than N^δ for some positive value of δ. For $u > N$ we need a slightly finer approximation, but the same principle is in operation. We set $B = 2u^{-1}$, $\sigma^2 = 4Nu^{-2}$ and $Y = \varepsilon p u^{\frac{1}{2}(\alpha^2-1)}$. Again, to apply Lemma 6.5

we must verify that $YB < 2p\sigma^2$: Comparing the two sides we find that it will certainly be true when $u^{\alpha^2+1} < N^2$. Calculating the probability we find an exponent of

$$\frac{1}{4}c\varepsilon^2 pN^{-1}u^{\alpha^2+1}, \tag{6.9}$$

which because of our estimation of p and the fact that $u > N$, is larger than N^δ for some $\delta > 0$.

When $u^{\alpha^2+1} \geq N^2$, we choose a small constant $0 < \eta < 1$ and set $t = \eta B^{-1}$. Using this in 6.2, we find that the exponent of the probability is

$$\begin{aligned} t^2 p\sigma^2 - tY &= \eta^2 pN - \frac{1}{2}\eta u\varepsilon p u^{\frac{1}{2}(\alpha^2-1)} \\ &\leq \eta^2 pN - \frac{1}{2}\eta u\varepsilon N^{1+\beta^2}N^{o(1)} \\ &\leq \eta^2 N - \frac{1}{2}\eta\varepsilon N^{2+\beta^2}N^{o(1)}. \end{aligned} \tag{6.10}$$

The inequality will then hold for the finite number of u which can be represented as $u = jN^{-2}$, $0 \leq j \leq N^4$, except on a set of measure $\exp(-N^{\delta_1})$, where $\delta_1 = \min\{\delta(u) | u = jN^{-2}; 0 \leq j \leq N^4\}$. Because the expression $\hat{\mu}_1(u) - \hat{\mu}_0(u)$ has derivative of at most 2, we interpolate between the fractions to find that for all u in the interval,

$$|\hat{\mu}_1(u) - \hat{\mu}_0(u)| < \varepsilon(1+u)^{\frac{1}{2}(\alpha^2-1)} + \frac{2}{N^2}$$

with probability at least $1 - \exp(-N^\delta)$, which still implies the statement of the lemma.

For those $u > N^2$, we find in inequality 6.9 an upper bound of

$$\eta^2 N - \frac{1}{2}\eta\varepsilon N^{3+\beta^2}N^{o(1)}, \tag{6.11}$$

which is independent of u and also tends to 0 as $N \to \infty$. Originally, this case was handled by the inequality

$$\|\hat{\mu}_1\| < 2u^{-1}p^{-1}\sum \xi_n < 4u^{-1}N = o(u^{-\frac{1}{2}}).$$

We fail to see how this leads to the required result. It can, however, be done as follows: We find the largest of the estimates 6.8 - 6.11, which gives us a certain upper bound on the probability of the extraordinary event. This upper bound tends to 0. The implication is, of course, that by enlarging N we can make the probability arbitrarily small. Thus follows the lemma. $\square$

We now proceed to the proof of Theorem 6.4.We let $\beta_1 = \beta - 2b^{-\frac{1}{2}}$ tend to α such that the difference is a sequence $\eta_1, \eta_2 = \eta^2, \eta_3 = \eta^3, \ldots$ converging to 0, where $\eta_1 < 1/2$. Now assume that the construction in the lemma has been accomplished on the interval $[0, 1]$, with $N = N_1$ such that the lemma holds except on a set of probability at most η_1. We then create a measure $\mu_2(A) = p_2^{-1}\xi_2(X)\mu_1(A)$, with p_2 and ξ_2 being the probability and characteristic function applicable to a difference (between α and β_1) of smaller than η_2, respectively. Note that p_2 here refers to the probability associated with a division of into N_2 intervals (since we use the number of intervals to approximate the probability), and therefore p_1 is still taken into account. Lemma 6.6 is not shown in [22] to lead to similar results for all n. How the iterated construction of μ_n (which is crucial to the theorem) is accomplished is not mentioned either.

The lemma can now be applied to show that

$$|\mu_2(u) - \mu_1(u)| < \frac{\varepsilon}{4}(1+u)^{\frac{1}{2}(\alpha^2-1)}$$

with probability approaching 1. The difference in applying the lemma with a "starting measure" of μ_1 as opposed to μ_0 is a factor $p_1^{-1}\xi_n$. We can factor out p_1 and bound the characteristic function by 1, and since p_1 is fixed (since N_1 is) we can adjust the ε to obtain the necessary inequality. We suppose that N_2 is chosen large enough so that the lemma holds except on a set of maximum measure $\eta_2 = \eta^2$.

A direct consequence of Lemmas 5.2 and 5.3 is that the total mass of the consecutive measures are bounded from above by 1 and from below by some constant c strictly larger than 0, for all N large enough. This follows because the proofs of the lemmas are still valid when the interval-ratio h^{β^2} is replaced by a term $h^{\beta^2}h^{o(1)}$. The constant c is absolute, thus we can be assured that none of the measures converge to zero. At each stage we then normalise the measures to maintain a total mass of 1 throughout. It is asserted in [22] that the measures have weight smaller than $1 + \eta + \cdots + \eta^n$ at each stage, but no argument supporting this is given.

Taking into account that the supports are nested as well, these measures converge to a measure μ on a subset of E_α. This satisfies

$$|\hat{\mu}_1(u) - \hat{\mu}_n(u)| < \varepsilon 2^{-n}(1+u)^{\frac{1}{2}(\alpha^2-1)}.$$

This measure has the required property, because

$$\begin{aligned} |\hat{\mu}(u) - 0| &\leq |\hat{\mu}(u) - \hat{\mu}_n(u)| + |\hat{\mu}_0 - 0| \\ &\leq \varepsilon(2^{-n} + \cdots + 2^{-2} + 1)(1+u)^{\frac{1}{2}(\alpha^2-1)} + |\hat{\mu}_0(u)| \end{aligned}$$

and therefore

$$\frac{|\hat{\mu}(u)|}{(1+u)^{\frac{1}{2}(\alpha^2-1)}} \leq 2\varepsilon + \frac{2u^{-1}}{(1+u)^{\frac{1}{2}(\alpha^2-1)}}.$$

This last expression clearly approaches 2ε as $u \to \infty$. We therefore have, for arbitrarily small ε and η, a measure μ which has the desired property except on a set of measure less than 2η. Here is where Kaufman concludes his proof of the theorem. We feel, however, that it is necessary to complete the argument as follows.

Our measure was determined by our choices of ε and η. These basically determine how large N should be chosen at each stage. The question now is whether different choices of (ε, η) yield different measures μ. Given a sequence $\{\varepsilon_i, \eta_i\}$, where both components tend to 0, we can find a sequence of measures $\mu(\varepsilon_i, \eta_i)$, each of which satisfies the same relations as the measure we have constructed above. We have obtained a sequence $\{N_i^1\}$ for the construction of the measure $\mu(\varepsilon_1, \eta_1)$. In the construction of $\mu(\varepsilon_2, \eta_2)$, we can assume that the sequence $\{N_i^2\}$ is a subsequence of $\{N_i^1\}$, since the size of the chosen N at each stage is really all that matters. Because of the nature of the construction, this implies that smaller η_i and ε_i simply assume that the same construction is started with a larger N_i, but will have the same limit. This implies that μ is indeed the required measure.

7 Appendix

As a short digression we will give a summary here of part of Orey and Taylor's proof that the α-rapid points of Brownian motion have dimension $1-\alpha^2$. This is a beautiful application of the power that Cantor-like sets may indeed posses (Mandelbrot [28] employs the term "Cantor dusts"), since the lower bound of dimension is determined through that of a Cantor-like subset. We will need the concept of a *Hausdorff measure function*. In our previous use of Hausdorff sums we merely raised the diameter of each set in the cover to a power α. This corresponds to a Hausdorff measure function $\psi(x) = x^\alpha$. We define the ψ-Hausdorff measure of a set a in exactly the same way as the α-Hausdorff measure in Chapter 1, except that the terms $|B|^\alpha$ in the sums are replaced by $\psi(|B|)$. The ψ-Hausdorff measure of A is denoted as $\psi - m(A)$.

Before we start the proof proper, we present a lemma concerning Cantor-like sets.

Lemma 7.1. *Suppose that ψ is a Hausdorff measure function and $c > 0$, $\delta > 0$. Let K be a Cantor-like set with representation*

$$K = \bigcap_{m=1}^{\infty} E_m, \quad E_{m+1} \subseteq E_m, \quad E_m = \bigcup_{i=1}^{M_m} I_{m,i},$$

where the $I_{m,i}$ $(1\ leqi \leq M_m)$ are disjoint closed subintervals of $[0,1]$. Then

$$\psi - m(K) > 0$$

if, for every interval $J \subseteq [0,1]$ with $|J| < \delta$, there is a finite integer $m(J)$ such that

$$M_m(J) \leq c\psi(|J|)M_m \quad \text{for } m \geq m(J), \tag{7.12}$$

where $M_m(J)$ denotes the number of intervals $I_{m,i}$ $(1 \leq i \leq M_m)$ contained in J.

This lemma can be used to show that $\psi - m(K) = 1$ for the classic Cantor triadic set, as shown in Chapter 3.

The easier part of the proof, which we consider first, considers the upper bound for the dimension. We define, as in Chapter 6,

$$S(a,b) = \max_{a \leq < s < t \leq b} |X(t) - X(s)|,$$

where X of course represents a Brownian motion. When $[a,b] = I$, we define $S(I)$ in the obvious way. We let $E(\alpha)$ $(\alpha > 0)$ denote the set of α-rapid points of X, as usual. Take $\alpha_2 < \alpha_1 < \alpha$ and consider the collection $\mathcal{I} = \mathcal{I}(\alpha_1)$ of intervals $I \subseteq [0,1]$ such that

$$S(I) > \alpha_1(2h \log h^{-1})^{\frac{1}{2}}, \quad |I| = h.$$

Using the estimate

$$\mathbf{P}\{S(I) > \lambda |I|^{\frac{1}{2}}\} = e^{-\frac{1}{2}\lambda^2(1+o(1))}$$

for the upper tail of $S(I)$, we can conclude that there is a $\delta = \delta(\alpha_2)$ such that

$$\mathbf{P}\{I \in \mathcal{I}\} < |I|^{\alpha_2^2}, \quad 0 < |I| < \delta.$$

Let $\mathcal{C}_n$ consist of all closed intervals of length $h_n = \exp -n/\log n$ and left-hand endpoints $ih/\log n$ $(i = 0, 1, 2, \ldots, [h_n^{-1} \log n])$. To any point $t \in E(\alpha)$ there corresponds a sequence of intervals $I_n = [t, t + u_n]$ such that $u_n \to 0$ and

$$S(I_n) > \frac{1}{2}(\alpha_1 + \alpha)(2u_n \log u_n^{-1})^{\frac{1}{2}}.$$

For small enough u_n, each such I_n is contained in one of the intervals $\mathcal{C}_m \cap \mathcal{I}$ for suitable m. Each point of $E(\alpha)$ is therefore covered infinitely often by intervals from the collection $\bigcup_{m=1}^{\infty} \mathcal{C}_{\Updownarrow} \cap \mathcal{I}$. The above estimate for the tail of S now yields

$$\mathbb{E}\{T_m\} \leq (h_m^{-1} \log m) h_m^{\alpha_2^2 - 1} \log m.$$

Thus, for each $\varepsilon > 0$,

$$\mathbf{P}\{T_m > h_m^{\alpha_2^2 - 1 - \varepsilon}\} \leq h_m^{\varepsilon} \log m.$$

By using the Borel-Cantelli lemma we can show that almost surely there is an integer $m_0 = m_0(\omega)$ such that

$$T_m \leq h_m^{\alpha_2^2 - 1 - \varepsilon}, \quad m \geq m_0.$$

Therefore

$$\sum T_m h_m^s < \infty, \quad s > 1 - \alpha_2^2 + \varepsilon$$

and $\dim E(\alpha) \leq 1 - \alpha_2^2 + \varepsilon$. Letting $\varepsilon \to 0$ and $\alpha_2 \to \alpha$ through countable sets, we find that

$$\dim E(\alpha) \leq 1 - \alpha^2.$$

To prove the opposite inequality, they consider a collection $\mathcal{I}$ of intervals $[u, v] \subseteq [0, 1]$ such that

$$X(v) - X(u) \geq \alpha \sqrt{2(v - u) \log (v - u)^{-1}},$$

where X of course represents a Brownian motion. The Lévy modulus (mentioned in Chapter 1) tells us that

$$|X(t) - X(s)| < 2\sqrt{|t - s| \log |t - s|^{-1}}$$

for $|s-t|$ small enough. Given some $\alpha_0 > \alpha$, this then guarantees the existence of some $b = b(\alpha, \alpha_0)$ such that for a sufficiently small interval $I = [u, v] \subseteq [0, 1]$,

$$X(v) - X(u) > \alpha_0 \sqrt{2(v-u)\log{(v-u)^{-1}}} \tag{7.13}$$

implies that $[t, v] \in \mathcal{I}$ for all $t \in I(b) = [u, u + b(v-u)] \subseteq [u, v]$. We can assume b to be the reciprocal of an integer. Now, suppose that ρ_m is the reciprocal of an integer, $\rho_{m+1} < b\rho_m$ and $b\rho_m/\rho_{m+1}$ is an integer for all integer m. We then let $\mathcal{F}_m$ be the class of all intervals of the form $[i\rho_m, (i+1)\rho_m]$, $i = 0, 1, 2, \ldots, p_m^{-1} - 1$ and let $\mathcal{F}_m^+$ denote the sequence of those intervals in $\mathcal{F}_m$ which satisfy (7.2) with α_0 replaced be α. Let

$$\mathcal{F}_m^+(b) = (I(b) : I \in \mathcal{F}_m^+).$$

A sparse subsequence of $\mathcal{F}_m^+(b)$ is then used to construct a Cantor-like subset of the set of α-rapid points, which may be quite sparse but still has large enough dimension.

Now, if J is a sub-interval of $[0, 1]$, we denote by $N_m(J)$ the random variables which count the number of intervals I of $\mathcal{F}_m^+$ contained in J and we let $N_m = N_m([0, 1])$. These variables have a binomial distribution, which leads to the following:

Lemma 7.2. *Given $\varepsilon > 0, \delta > 0$, then there almost surely exists an integer m_0 such that*

$$|N_m(J) - E\{N_m(J)\}| < \varepsilon E\{N_m(J)\}$$

for all $J \subseteq [0, 1]$ such that $|J| \geq \delta$, and all $m \geq m_0(\varepsilon, \delta)$.

This takes care of the intervals that are large compared to ρ_m, but does not quite work for the small ones. For these we use the following:

Lemma 7.3. *Given $\beta_0 < \beta = 1 - \alpha_0^2$, there is an absolute constant c such that almost surely there is an m_1 such that*

$$N_m(J) \leq c|J|^{\beta_0} N_m$$

for all $J \subseteq [0, 1]$ with $|J| \leq \rho_m$, $m \geq m_1$.

Combining these we have constructed a Cantor-like set which has dimension close to $1 - \alpha^2$. By then having α_0 converge to α and ε converge to zero, the theorem is proved. It should be noted that these are significant steps by themselves, since convergence in the area of dimension is not always clear cut. For instance, a nested sequence of Cantor-like sets of dimension 1 can converge (by taking the intersection) to a set of dimension 0.

References

[1] S. Albeverio, J.E. Fenstad, R. Hoegh-Krohn, T. Lindstrom, "Nonstandard Methods in Stochastic Analysis and Mathematical Physics", Academic Press, 1986.

[2] R.M. Anderson, A nonstandard representation for Brownian motion and Itô integration, *Israel J. Math.* **25** (1976).

[3] A.S. Besicovitch, Sets of fractional dimensions (IV): On rational approximation to real numbers, *J. Lond. Math. Soc.* **9** (1934), 126-131.

[4] P. Billingsley, "Convergence of Probability Measures", Wiley, New York, 1968.

[5] C. Bluhm, On a theorem of Kaufman: Cantor-type construction of linear fractal Salem sets, Ark. Mat., **36** (1998), 307-316.

[6] G. Cantor and R. Dedekind, "Briefwechsel", Ed. E. Noether and J. Cavallès, Paris: Hermann, 1937.

[7] NJ Cutland, "Loeb Measures in Practice: Recent Advances", Springer-Verlag, 1997.

[8] M.D. Donsker, An invariance principle for certain probability limit theorems, *Memoirs of the American Mathematical Society* **6** (1951).

[9] A. Dvoretsky, P. Erdős, S. Kakutani, Nonincreasing everywhere of the Brownian motion process, *Proc. Fourth Berkeley Symp. on Math. Stat. and Probability II*, 103-116, Univ. of Cali. Press, 1961.

[10] W. Feller, "An Introduction to Probability Theory and Its Applications", Vol. I, 3d ed., John Wiley & Sons, Inc., 1968.

[11] W.L. Fouché, Arithmetical representations of Brownian motion 1, *J. Symbolic Logic* **65** (2000), 421-442.

[12] W.L. Fouché, Local properties of oscillations of high Kolmogorov complexity, preprint.

[13] O. Frostman, Potentiel d'équilibre et capacité des ensembles, Thèse, Lund, 1935.

[14] M. Gatesoupe, Sur un théorème de R. Salem, *Bull. Soc. Math. Fr.* **91** (1967), 125-127.

[15] G.H. Hardy and E.M. Wright, "An Introduction to the Theory of Numbers", Oxford University Press, 4th ed., 1971.

[16] F. Hausdorff, Dimension und äusseres Mass, *Mathematische Annalen* **79** (1919), 157-179.

[17] F. Hausdorff, “Grundzüge der Mengenlehre”, Leipzig, 1914. Reprinted by Chelsea, New York, 1941.

[18] T. Hida, “Brownian Motion”, Springer-Verlag, 1980.

[19] K. Itô and H.P. McKean, Jr., “Diffusion Processes”, Springer-Verlag, 1974.

[20] V. Jarnik, Zur metrischen Theorie der diophantischen Approximation, *Prace Mat.-Fiz.* **36** (1928/29), 91-106.

[21] J-P. Kahane, “Some Random Series of Functions” 2nd ed., Cambridge University Press, 1985.

[22] R. Kaufman, Large increments of Brownian motion, *Nagoya Math. J.* **56** (1974), 139-145.

[23] R. Kaufman, On the theorem of Jarnik and Besicovitch, *Acta Arithm.* **39** (1981), 265-267.

[24] A.S. Kechris and A. Louveau, “Descriptive Set Theory and the Structure of Sets of Uniqueness”, Cambridge University Press, 1987.

[25] A.I. Khintchine, Über dyadische Brüche, *Mathematische Zeitschrift* **18** (1923), 109-116.

[26] P. Lévy, “Théorie de l’Addition des Variables Aleatoires”, 2nd ed., Paris, Gauthier-Villars, 1954.

[27] P.A. Loeb, Conversion from nonstandard to standard measure spaces and applications in probability theory, *Trans. Amer. Math. Soc.* **211** (1975).

[28] B. Mandelbrot, “The Fractal Geometry of Nature”, Freeman, 1983.

[29] P. Mattila, “Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability”, Cambridge University Press, 1995.

[30] R. Alain, “ Nonstandard Analysis”, John Wiley & Sons, 1985.

[31] S. Orey and S.J. Taylor, How often on a Brownian path does the law of iterated logarithm fail?, *Proc. Lond. Math. Soc.* **28**(3) (1974), 174-192.

[32] E. Perkins, A global intrinsic characterisation of Brownian local time, *Ann. Prob.* **9** (1981).

[33] W. Rudin, Fourier-Stieltjes transforms of measures on independent sets, *Bull. Amer. Math. Soc.* **66** (1960), 199-202.

[34] R. Salem, On singular monotonic functions whose spectrum has a given Hausdorff dimension, *Arkiv. Mat.* **1** (1950), 353-365.

[35] H. Steinhaus, Sur les distances des points des ensembles de mesure positive, *Fundementa Mathematicae* **1** (1920), 93-104.

[36] Wattenberg, Nonstandard measure theory — Hausdorff measure, *Proc. AMS* **65** (1977), 326-331.

[37] N. Wiener, Differential space, *J. Math. Phys.* **2** (1923), 131-174.